Multiple comparison procedures for high-dimensional data and their robustness under non-normality

Sho Takahashi[a*], Masashi Hyodo[b†] Takahiro Nishiyama[c], and Tatjana Pavlenko[d]

[a] *Department of Mathematical Information Science, Graduate School of Science, Tokyo University of Science*
1-3, Kagurazaka, Shinjuku-ku, Tokyo, 162-8601 JAPAN

[b] *Graduate School of Economics, The University of Tokyo*
7-3-1, Hongo, Bunkyo-ku, Tokyo, 113-0033 JAPAN

[c] *Department of Mathematical Information Science, Faculty of Science, Tokyo University of Science*
1-3, Kagurazaka, Shinjuku-ku, Tokyo, 162-8601 JAPAN

[d] *Department of Mathematics, KTH Royal Institute of Technology*
106-91, Stockholm, Sweden

Abstract

This paper analyzes whether procedures for multiple comparison derived in Hyodo et al. (2012) work for an unbalanced case and under non-normality. We focus on pairwise multiple comparisons and comparison with a control among mean vectors, and show that the asymptotic properties of these procedures remain valid in unbalanced high-dimensional setting. We also numerically justify that the derived procedures are robust under non-normality, i.e., the coverage probability of these procedures can be controlled with or without the assumption of normality of the data.

Key words and phrases: Asymptotic expansion; Bonferroni's inequality; Comparison with a control; Dempster trace criterion; High-dimensional data; Monte Carlo Simulation; Pairwise comparison; Power, Robustness.

*Corresponding author. Tel.: +81-3-3260-4271; Fax: +81-3-3260-4293.
E-mail address: `j1410704@ed.tus.ac.jp` (S. Takahashi)
†Research Fellow of the Japan Society for the Promotion of Science

1 Introduction and motivation

In this paper, we consider multiple comparisons and comparisons with a control among mean vectors in the unbalanced case, i.e., assuming unequal sample sizes. Let $\boldsymbol{x}_j^{(i)}$ $(j = 1, 2, \ldots, N_i,\ i = 1, 2, \ldots, k)$ be independently distributed as the p-dimensional normal distribution with mean vector $\boldsymbol{\mu}^{(i)}$ and common covariance matrix Σ. Let the i-th sample mean vector and the pooled sample covariance matrix be

$$\overline{\boldsymbol{x}}^{(i)} = \frac{1}{N_i}\sum_{j=1}^{N_i}\boldsymbol{x}_j^{(i)}, \quad S = \frac{1}{m}\sum_{i=1}^{k}\sum_{j=1}^{N_i}(\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})(\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})',$$

respectively, where $m = \sum_{i=1}^{k} N_i - k$.

In general, the simultaneous confidence intervals for pairwise multiple comparisons among mean vectors with the confidence level $1-\alpha$ are given by

$$\begin{aligned} \boldsymbol{a}'(\boldsymbol{\mu}^{(i)} - \boldsymbol{\mu}^{(j)}) \quad &\in \quad \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(j)}) \pm t_{\mathrm{p}}\sqrt{w_{ij}\boldsymbol{a}'S\boldsymbol{a}}\right], \\ &\qquad \forall\boldsymbol{a} \in \boldsymbol{R}^p - \{\boldsymbol{0}\}, \quad 1 \le i < j \le k, \end{aligned}$$

where $w_{ij} = 1/N_i + 1/N_j$, $\boldsymbol{R}^p - \{\boldsymbol{0}\}$ is the set of any nonnull real p-dimensional vectors, and $t_{\mathrm{p}}^2 \equiv t_{\mathrm{p}}^2(\alpha)$ is the upper 100α percentile of the distribution of $T^2_{\max\cdot\mathrm{p}}$ defined by

$$\Pr\{T^2_{\max\cdot\mathrm{p}} > t_{\mathrm{p}}^2\} = \alpha,$$

where

$$\begin{aligned} T^2_{\max\cdot\mathrm{p}} &= \max_{1\le i<j\le k}\{T_{ij}^2\}, \\ T_{ij}^2 &= w_{ij}^{-1}(\boldsymbol{y}^{(i)} - \boldsymbol{y}^{(j)})'S^{-1}(\boldsymbol{y}^{(i)} - \boldsymbol{y}^{(j)}), \\ \boldsymbol{y}^{(i)} &= \overline{\boldsymbol{x}}^{(i)} - \boldsymbol{\mu}^{(i)}, \quad i = 1, 2, \ldots, k. \end{aligned}$$

By analogy with above, in the case of comparisons with a control, letting the first population be a control, the simultaneous confidence intervals are given by

$$\begin{aligned} \boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(j)}) \quad &\in \quad \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(j)}) \pm t_{\mathrm{c}}\sqrt{w_{1j}\boldsymbol{a}'S\boldsymbol{a}}\right], \\ &\qquad \forall\boldsymbol{a} \in \boldsymbol{R}^p - \{\boldsymbol{0}\}, \quad 2 \le j \le k, \end{aligned}$$

and the value t_{c} (> 0) is chosen to satisfy

$$\Pr\{T^2_{\max\cdot\mathrm{c}} > t_{\mathrm{c}}\} = \alpha,$$

where

$$\begin{aligned} T^2_{\max\cdot\mathrm{c}} &= \max_{2\le j\le k}\{T^2_{1j}\}, \\ T^2_{1j} &= w^{-1}_{1j}(\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(j)})' S^{-1}(\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(j)}). \end{aligned}$$

In order to construct simultaneous confidence intervals, it is required to obtain the upper percentiles of $T^2_{\max}$-type statistics, i.e., t_{p} and t_{c}. In general, it is difficult to obtain them exactly even under normality. The use of Bonferroni's inequality to obtaining approximate upper percentiles are discussed by e.g., Seo (1995), Seo and Siotani (1992). These results are derived for the standard asymptotic setting, i.e., assuming that p is fixed and is much smaller than N_i.

However, recently, high-dimensional data have been increasingly encountered in many applications of statistics and most prominently in biological and financial studies. It is well known that when the dimension is larger than the total sample size, the sample covariance matrix becomes singular, and hence it will be impossible to define Hotelling's $T^2_{\max}$-type statistic. To tackle this problem efficiently, the Dempster trace criterion for one and two sample can be used.

The technique considered in the current study develops results derived in Dempster (1958, 1960). The similar approach for multivariate linear hypothesis has been also discussed by Fujikoshi et al. (2004), Himeno (2007) and many other authors.

To adjust the high-dimensional setting to the unbalanced case, we consider the following asymptotic framework:

$$\begin{aligned} \mathrm{A1} &: \; m \to \infty, \quad p \to \infty, \quad p/m \to \gamma \in (0, \infty), \\ \mathrm{A2} &: \; 0 < \lim_{p\to\infty} \mathrm{tr}\Sigma^i/p < \infty, \quad i = 1, 2, \ldots, 8. \end{aligned}$$

Using conventional terminology, from now on we will refer to the assumption A1 as (m, p)-*asymptotics.*

In Hyodo et al. (2012), the balanced case is considered and the following $D_{\max}$-type test statistic

$$\begin{aligned} D_{\max\cdot\mathrm{p}} &= \max_{1\le i<j\le k}\{D_{ij}\}, \\ D_{ij} &= \frac{p}{\widehat{\sigma}}\left\{\frac{W_{ij}}{\mathrm{tr}S}-1\right\} = \frac{p}{\widehat{\sigma}}\left\{\frac{(\boldsymbol{y}^{(i)}-\boldsymbol{y}^{(j)})'(\boldsymbol{y}^{(i)}-\boldsymbol{y}^{(j)})}{w_{ij}\mathrm{tr}S}-1\right\} \end{aligned} \tag{1}$$

is proposed for pairwise comparisons. Further, for comparisons with a control, the following statistic

$$\begin{aligned} D_{\max\cdot\mathrm{c}} &= \max_{2\le j\le k}\{D_{1j}\}, \\ D_{1j} &= \frac{p}{\widehat{\sigma}}\left\{\frac{W_{1j}}{\mathrm{tr}S}-1\right\} = \frac{p}{\widehat{\sigma}}\left\{\frac{(\boldsymbol{y}^{(1)}-\boldsymbol{y}^{(j)})'(\boldsymbol{y}^{(1)}-\boldsymbol{y}^{(j)})}{w_{1j}\mathrm{tr}S}-1\right\} \end{aligned} \tag{2}$$

is derived, where $\widehat{\sigma} = \sqrt{2p\widehat{c}_2/\widehat{c}_1^2}$ and $\widehat{c}_i$'s are the unbiased and consistent estimators of $c_i = \mathrm{tr}\Sigma^i/p$. Further, the Bonferroni approximations for the upper percentiles of these statistics are derived.

This study gives an extension of the results by Hyodo et al. (2012) to the unbalanced case. We show that the consistency and asymptotic normality of D_{ij} in (1) and D_{1j} in (2), remain valid for the unbalanced case. For the procedures derived in Hyodo et al. (2012), the coverage probability can be controlled provided the assumption that the data are normally distributed. However, so far no satisfactory test is available to ascertain the multivariate normality of the data when $N_i \le p$. Hence, it would be desirable to derive such multiple comparison procedures for which the coverage probability can be controlled with or without the assumption of normality of the data, that is procedures that robust under non-normality. Our objective in this study is to numerically justify that the extended comparison procedures are robust for the model described above.

The rest of the paper is organized as follows: In Section 2, the main asymptotic properties are shown to be valid for the unbalanced case. Further, the Bonferroni approximations of the upper 100α percentiles of $D_{\max}$-type statistics is derived for pairwise comparisons in Section 2, and for for comparisons with a control in Section 3, respectively. Section 4 provides numerical examination of the performance accuracy of the extended procedures. To justify the robustness of these procedures, we compare the power of $D_{\max}$-type statistics with that of the data generated from a number of non-normal distributions. At last, we provide some concluding remarks. In Section 3, the approximation is given for comparisons with a control.

2 Pairwise comparisons in high-dimensional framework – unbalanced case

Consider the following simultaneous confidence intervals for mean vectors:

$$\begin{aligned} \boldsymbol{a}'(\boldsymbol{\mu}^{(i)} - \boldsymbol{\mu}^{(j)}) \quad &\in \quad \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(j)}) \pm d_{\mathrm{p}}\sqrt{w_{ij}(\mathrm{tr}S)\boldsymbol{a}'\boldsymbol{a}}\right], \\ &\forall \boldsymbol{a} \in \boldsymbol{R}^p - \{\boldsymbol{0}\}, \quad 1 \le i < j \le k, \end{aligned}$$

where $d_{\mathrm{p}}^2 = 1 + (\widehat{\sigma}/p)z_{\mathrm{p}}$ and $z_{\mathrm{p}} \equiv z_{\mathrm{p}}(\alpha)$ is the upper 100α percentile of $D_{\max\cdot\mathrm{p}}$ statistic. In high-dimensional framework, it is difficult to give the exact value of z_{p}. In this study, we derive the approximation for z_{p} using Bonferroni inequality.

By applying Bonfferoni's inequality to $\Pr\{D_{\max\cdot\mathrm{p}} > z_{\mathrm{p}}\}$, we get

$$\Pr\{D_{\max\cdot\mathrm{p}} > z_{\mathrm{p}}\} < \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \Pr\{D_{ij} > z_{\mathrm{p}}\}.$$

We then define the Bonferroni approximation for z_{p} as such $z_{1\cdot\mathrm{p}}$ which satisfies

$$\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \Pr\{D_{ij} > z_{1\cdot\mathrm{p}}\} = \alpha$$

for a given α. It is well known that $z_{1\cdot\mathrm{p}}$ essentially overestimates the upper 100α percentile of $D_{\max\cdot\mathrm{p}}$. In order to get a corrected estimator of this percentile, we derive the Cornish-Fisher asymptotic expansion of $z_{1\cdot\mathrm{p}}$. For this expansion, we need two auxiliary results, stated in Lemma 1 and Theorem 2 below.

Lemma 1. *Under the high-dimensional asymptotic framework A1 and assumption A2, the unbiased and (m,p)-consistent estimators of c_i's $(i = 1, 2, 3, 4)$ can be obtained as*

$$
\begin{aligned}
\widehat{c}_1 &= \frac{\mathrm{tr}S}{p}, \\
\widehat{c}_2 &= \frac{m^2}{(m+2)(m-1)p}\left\{\mathrm{tr}S^2 - \frac{(\mathrm{tr}S)^2}{m}\right\}, \\
\widehat{c}_3 &= \frac{m^4}{(m+4)(m+2)(m-1)(m-2)p}\left\{\mathrm{tr}S^3 - \frac{3}{m}\mathrm{tr}S^2\mathrm{tr}S + \frac{2}{m^2}(\mathrm{tr}S)^3\right\}, \\
\widehat{c}_4 &= \frac{1}{p}\left\{b_1\mathrm{tr}S^4 + b_2\mathrm{tr}S^3\mathrm{tr}S + b_3(\mathrm{tr}S^2)^2 + b_4\mathrm{tr}S^2(\mathrm{tr}S)^2 + b_5(\mathrm{tr}S)^4\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
b_1 &= \frac{m^5(m^2+m+2)}{(m+6)(m+4)(m+2)(m+1)(m-1)(m-2)(m-3)}, \\
b_2 &= -\frac{4m^4(m^2+m+2)}{(m+6)(m+4)(m+2)(m+1)(m-1)(m-2)(m-3)}, \\
b_3 &= -\frac{m^4(2m^2+3m-6)}{(m+6)(m+4)(m+2)(m+1)(m-1)(m-2)(m-3)}, \\
b_4 &= \frac{2m^4(5m+6)}{(m+6)(m+4)(m+2)(m+1)(m-1)(m-2)(m-3)}, \\
b_5 &= -\frac{m^3(5m+6)}{(m+6)(m+4)(m+2)(m+1)(m-1)(m-2)(m-3)},
\end{aligned}
$$

and

$$
\mathrm{Var}[\widehat{c}_i] = O(m^{-2}).
$$

Proof. The proof follows from the results by Srivastava (2005) and Hyodo et al. (2012).

□

Theorem 2. *Under the high-dimensional asymptotic framework A1 and assumption A2, it holds that*

$$
D_{ij} \xrightarrow{d} N(0,1),
$$

where "$\xrightarrow{d}$" denotes convergence in distribution.

Proof. The proof follows by applying the technique by Fujikoshi et al. (2004) for D_{ij}. That is, letting $q = 1$ in Fujikoshi et al. (2004), D_{ij} corresponds to $\widetilde{T}_D/\widehat{\sigma}_D$. Therefore, the desired result is obtained. □

From the numerical results in Hyodo et al. (2012), it can be observed that the approximation by using Theorem 2 is not conservative, i.e., $z_{1\cdot\mathrm{p}} \leq z_{\mathrm{p}}$. Therefore, we next consider improving on the accuracy of the approximation using asymptotic expansion. In the same way as Hyodo et al. (2012), the characteristic function $C(t) = \mathrm{E}[\exp(\mathrm{i}tD_{ij})]$ is expanded as

$$C(t) = \exp\left\{\frac{(\mathrm{i}t)^2}{2}\right\}\left\{1 + \frac{1}{\sqrt{p}}d_3(\mathrm{i}t)^3 + \frac{1}{p}\{d_4(\mathrm{i}t)^4 + d_6(\mathrm{i}t)^6\} + \frac{1}{m}d_2(\mathrm{i}t)^2\right\} + O(p^{-\frac{3}{2}}),$$

where $\mathrm{i} = \sqrt{-1}$ and

$$d_2 = \frac{1}{2}, \quad d_3 = \frac{\sqrt{2}c_3}{3\sqrt{c_2^3}}, \quad d_4 = \frac{c_4}{2c_2^2}, \quad d_6 = \frac{c_3^2}{9c_2^3}.$$

Inverting $C(t)$, we obtain

$$\begin{aligned}\Pr\{D_{\ell m} \leq z\} &= \Phi(z) - \phi(z)\left\{\frac{1}{\sqrt{p}}\left(\frac{\sqrt{2}c_3}{3\sqrt{c_2^3}}\right)h_2(z) + \frac{1}{p}\left(\frac{c_4}{2c_2^2}h_3(z) + \frac{c_3^2}{9c_2^3}h_5(z)\right)\right.\\ &\quad \left. + \frac{1}{2m}h_1(z)\right\} + O(p^{-\frac{3}{2}}),\end{aligned}$$

where $\Phi(z)$ and $\phi(z)$ are the distribution function and the density function of the standard normal distribution, respectively, and $h_i(z)$'s are the Hermite polynomials given by

$$h_1(z) = z, \quad h_2(z) = z^2 - 1, \quad h_3(z) = z^3 - 3z, \quad h_5(z) = z^5 - 10z^3 + 15z.$$

Further, by applying Cornish-Fisher expansion to $z_{1\cdot\mathrm{p}}$, we get the Bonferroni approximation as it is stated in the following theorem.

Theorem 3. *Under the high-dimensional asymptotic framework A1 and assumption A2, the Bonferroni approximation is given by*

$$\begin{aligned}z_{1\cdot\mathrm{p}} &= z_{\alpha_\mathrm{p}} + \frac{1}{\sqrt{p}}\left(\frac{\sqrt{2}c_3}{3\sqrt{c_2^3}}\right)(z_{\alpha_\mathrm{p}}^2 - 1) + \frac{1}{p}\left\{\frac{c_4}{2c_2^2}z_{\alpha_\mathrm{p}}(z_{\alpha_\mathrm{p}}^2 - 3) - \frac{2c_3^2}{9c_2^3}z_{\alpha_\mathrm{p}}(2z_{\alpha_\mathrm{p}}^2 - 5)\right\}\\ &\quad + \frac{1}{2m}z_{\alpha_\mathrm{p}} + O(p^{-\frac{3}{2}}),\end{aligned} \tag{3}$$

where z_α is the upper 100α percentile of the standard normal distribution, $\alpha_\mathrm{p} = \alpha/K$ and $K = k(k-1)/2$.

Proof. Similar to Hyodo et al. (2012), we can show the above theorem although the sample sizes are unequal. □

Further, by replacing the unknown parameters in (3) with their unbiased and (m,p)-consistent estimators given in Lemma 1, the following approximation of $z_{1\cdot \mathrm{p}}$ is obtained.

$$\begin{aligned}\widehat{z}_{1\cdot\mathrm{p}} &= z_{\alpha_\mathrm{p}} + \frac{1}{\sqrt{p}}\left(\frac{\sqrt{2}\widehat{c}_3}{3\sqrt{\widehat{c}_2^3}}\right)(z_{\alpha_\mathrm{p}}^2 - 1) + \frac{1}{p}\left\{\frac{\widehat{c}_4}{2\widehat{c}_2^2}z_{\alpha_\mathrm{p}}(z_{\alpha_\mathrm{p}}^2 - 3) - \frac{2\widehat{c}_3^2}{9\widehat{c}_2^3}z_{\alpha_\mathrm{p}}(2z_{\alpha_\mathrm{p}}^2 - 5)\right\} \\ &\quad + \frac{1}{2m}z_{\alpha_\mathrm{p}}.\end{aligned}$$

3 Approximation for comparisons with a control

The simultaneous confidence intervals for comparisons with a control are given by

$$\begin{aligned}\boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(j)}) &\in \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(j)}) \pm d_\mathrm{c}\sqrt{w_{1j}(\mathrm{tr}S)\boldsymbol{a}'\boldsymbol{a}}\right], \\ &\qquad \forall \boldsymbol{a} \in \boldsymbol{R}^p - \{\boldsymbol{0}\}, \quad 2 \le j \le k,\end{aligned}$$

where $d_\mathrm{c}^2 = 1 + (\widehat{\sigma}/p)z_\mathrm{c}$ and $z_\mathrm{c} \equiv z_\mathrm{c}(\alpha)$ is the upper 100α percentile of $D_{\max\cdot\mathrm{c}}$ statistic.

In a similar manner to $D_{\max\cdot\mathrm{p}}$, we obtain the following theorem on the Bonferroni approximation for z_c.

Theorem 4. *Under the high-dimensional and asymptotic frameworks A1 and A2, the Bonferroni approximation is given by*

$$\begin{aligned}z_{1\cdot\mathrm{c}} &= z_{\alpha_\mathrm{c}} + \frac{1}{\sqrt{p}}\left(\frac{\sqrt{2}c_3}{3\sqrt{c_2^3}}\right)(z_{\alpha_\mathrm{c}}^2 - 1) + \frac{1}{p}\left\{\frac{c_4}{2c_2^2}z_{\alpha_\mathrm{c}}(z_{\alpha_\mathrm{c}}^2 - 3) - \frac{2c_3^2}{9c_2^3}z_{\alpha_\mathrm{c}}(2z_{\alpha_\mathrm{c}} - 5)\right\} \\ &\quad + \frac{1}{2m}z_{\alpha_\mathrm{c}} + O(p^{-\frac{3}{2}}),\end{aligned}$$

where $\alpha_\mathrm{c} = \alpha/(k-1)$.

Proof. Using the similar technique as in Theorem 3, we can show the above theorem.
□

4 Numerical examinations

4.1 Accuracy of approximations

We conduct Monte Carlo simulation with $\ell = 100000$ replications. The selected parameter values are $\alpha = 0.05$ and $p = 30, 60, 90, 120, 150, 200$. The sample sizes are set as $(N_1, N_2, N_3) = (20, 20, 20), (30, 20, 10), (40, 10, 10)$ when $k = 3$. Also let $N_{12} \equiv N_1 = N_2$, $N_{34} \equiv N_3 = N_4$ and $N_{56} \equiv N_5 = N_6$ when $k = 6$, then the sample sizes are set as $(N_{12}, N_{34}, N_{56}) = (10, 10, 10), (15, 10, 5), (20, 5, 5)$. The covariance structures are as follows: $\Sigma_1 = I_p$, $\Sigma_2 = (0.2^{|i-j|})$ and $\Sigma_3 = (0.5^{|i-j|})$. Attained confidence level (i.e., empirical coverage probability) of $D_{\max}$-type statistics can be numerically evaluated as

$$\begin{aligned} Q_{\mathrm{p}}(x) &= \frac{\sharp\{D_{\max\cdot\mathrm{p}} < x\}}{\ell}, \\ Q_{\mathrm{c}}(x) &= \frac{\sharp\{D_{\max\cdot\mathrm{c}} < x\}}{\ell}. \end{aligned}$$

Tables 1 and 2 list the simulated value z_{p}^* and the attained confidence level $Q_{\mathrm{p}}(x)$ for pairwise comparisons. It should be noted that $Q_{\mathrm{p}}(\widehat{z}_{1\cdot\mathrm{p}}) \geq 1-\alpha$. $\widehat{z}_{1\cdot\mathrm{p}}$ has a tendency to be more conservative as the number of populations increases. z_{p}^* larger for is balanced case and $Q_{\mathrm{p}}(\widehat{z}_{1\cdot\mathrm{p}})$ turns out to be more conservative for the unbalanced case.

Tables 3 and 4 list the simulated value z_{c}^* and the attained confidence level $Q_{\mathrm{c}}(x)$ for comparisons with a control. In contrast with pairwise comparisons, the balanced case is more conservative than the unbalanced cases.

4.2 Robustness of the proposed procedures under non-normality

Firstly, the robustness for the distribution is investigated by Monte Carlo simulation. We compare with the following five distributions:

D1: the multivariate normal distribution,

D2: the multivariate t distribution with 7 degrees of freedom, D3: the ε-contaminated normal distribution ($\kappa = 1.78$),

D4: the ε-contaminated normal distribution ($\kappa = 3.24$),

D5: the multivariate skew-normal distribution ($\delta = \delta_j = 0.5$, $j = 1, 2, \dots, p$).

It should be noted that D1–D4 belong to the class of elliptical distributions and symmetric distributions, whereas D5 represents the case of asymmetric distribution. Parameters are the same in the setup in Subsection 4.1.

Table 5 lists the attained confidence level for pairwise comparisons. When the sample sizes are balanced, all values are greater than or equal to 0.95. The values of D2–D4 is large comparing that of D1. When $(N_1, N_2, N_3) = (30, 20, 10)$, it can be observed the same tendency as the balanced case. However, the degree of conservativeness of D2–D4 is smaller than that of D1. Also when $(N_1, N_2, N_3) = (40, 10, 10)$, D1, D3 and D5 are greater than or equal to 0.95, however, D2 and D3 are less than 0.95. Further, D1 and D5 are just slightly effected by whether the sample sizes are balanced or unbalanced, in contrast to D2–D4 which are strongly influenced by the sample sizes. In particular, D2 can be expected to be very sensitive to high-dimensions due to heavy tail of the multivariate t distribution with low degrees of freedom.

Table 6 lists the attained confidence level for comparisons with a control. Almost all the values are greater than or equal to 0.95 when $(N_1, N_2, N_3) = (20, 20, 20), (30, 20, 10)$, in contrast, almost all the values are less than or equal to 0.95 when $(N_1, N_2, N_3) = (40, 10, 10)$.

From both Table 5 and Table 6, we see that the variability of the power results and attained confidence level is most pronounced for D2. Therefore, our procedures could be sensitive to large p when the distribution underlying the data has heavy tails.

Further, we numerically evaluate the power of the test. We re-express simultaneous confidence intervals as the following hypotheses testing problem:

$$H_{ij} : \boldsymbol{\mu}^{(i)} = \boldsymbol{\mu}^{(j)} \quad \text{vs.} \quad A_{ij} : \boldsymbol{\mu}^{(i)} \neq \boldsymbol{\mu}^{(j)}, \quad i < j.$$

Then the performance property of $D_{\max}$-type statistics is studied using a series of power simulations. Let $\alpha = 0.05$, $k = 3$, and mean vectors and covariance matrix are as follows: $\boldsymbol{\mu}^{(1)} = (\theta \mathbf{1}'_{p_1}, \mathbf{0}')'$ and $\boldsymbol{\mu}^{(2)} = \boldsymbol{\mu}^{(3)} = \mathbf{0}$, $\theta = 0.1, 0.3, 0.5, 0.7, 0.9$, $r = p_1/p = 0.25, 0.50, 0.75, 1.00$, and $\Sigma = I_p$. We set $p = 60$ and vary sample sizes as above and calculate the empirical power as

$$\begin{aligned}
\widehat{\beta}_{\mathrm{p}} &= \frac{\sharp\{D_{12} > \widehat{z}_{1\cdot\mathrm{p}},\ D_{13} > \widehat{z}_{1\cdot\mathrm{p}},\ D_{23} < \widehat{z}_{1\cdot\mathrm{p}} \text{ under } A_{12},\ A_{13},\ H_{23}\}}{\ell}, \\
\widehat{\beta}_{\mathrm{c}} &= \frac{\sharp\{D_{12} > \widehat{z}_{1\cdot\mathrm{c}},\ D_{13} > \widehat{z}_{1\cdot\mathrm{c}} \text{ under } A_{12},\ A_{13}\}}{\ell},
\end{aligned}$$

with $\widehat{\beta}_{\mathrm{p}}$ and $\widehat{\beta}_{\mathrm{c}}$ denoting the power of pairwise comparisons and that of comparisons with a control, respectively.

Table 7 lists the power for pairwise comparisons. The power of the test becomes greater as both r and θ increase. Further, it can be observed that the balanced case is the most powerful for all choices of θ and r.

Table 8 lists the power for comparisons with a control, showing the same tendency as in the pairwise case.

5 Concluding remarks and recommendations

The paper provides an extension of the results by Hyodo et al. (2012) and investigation of the robustness of the extended multiple comparison procedures under non-normality. Like that of Hyodo et al. (2012) our procedures are built in upon the Dempster trace criterion and focus on high-dimensional case that allow $N_i > p$. Unlike Hyodo et al. (2012), we look on the unbalanced case and show that consistency and asymptotic normality hold in the adjusted high-dimensional asymptotic framework. We suspect the asymptotic distribution of terms like D_{ij} in (1) and D_{1j} in (2) to be difficult to derive under the non-normality assumption and leave this question for the future work. However, our simulations indicate that the extended procedures in (1) and (2), appear to perform well for a number of non-normal distributions, and hence are robust under non-normality.

We thereby can recommend the use of $D_{\max}$-type statistics for both pairwise comparisons and comparisons with a control for the unbalanced case with very small sample sizes and very high-dimensionality. The best performance is achieved when the distribution underling the data is not heavy tailed one.

Acknowledgements

The research of the second author was partly supported by Japan Society for the Promotion of Science (JSPS). The research of the third author was partly supported by grant from The Royal Swedish Academy of Sciences, G.S. Magnuson's fond. The research of the forth author was partly supported by grant from Swedish Research Council (421-2008-1966).

References

Dempster, A. P. (1958). A high dimensional two samples significance test. *Annals of Mathematical Statistics* **29**, 995–1010.

Dempster, A. P. (1960). A significance test for separation of two highly multivariate small samples. *Biometrics* **16**, 41–50.

Fujikoshi, Y., Himeno, T. and Wakaki, H. (2004). Asymptotic results of a high dimensional MANOVA test and power comparison when the dimension is large compared to the sample size. *Journal of the Japan Statistical Society* **34**, 19–26.

Himeno, T. (2007). Asymptotic expansions of the null distributions for the Dempster trace criterion. *Hiroshima Mathematical Journal* **37**, 431–454.

Hyodo, M., Takahashi, S. and Nishiyama, T. (2012). Multiple comparisons among mean vectors when the dimension is larger than the total sample size. Submitted.

Seo, T. (1995). Simultaneous confidence procedures for multiple comparisons of mean vectors in multivariate normal populations. *Hiroshima Mathematical Journal* **25**, 387–422.

Seo, T. and Siotani, M. (1992). The multivariate studentized range and its upper percentiles. *Journal of the Japan Statistical Society* **22**, 123–137.

Siotani, M. (1959). The extreme value of the generalized distance of the individual points in the multivariate normal sample. *Annals of the Institute of Statistical Mathematics* **10**, 183–208.

Srivastava, M. S. (2005). Some tests concerning the covariance matrix in high dimensional data. *Journal of the Japan Statistical Society* **35**, 251–272.

Table 1: The attained confidence level for pairwise comparisons when $k = 3$.

				Σ_1			Σ_2			Σ_3		
N_1	N_2	N_3	p	z^*_{p}	$Q_{\mathrm{p}}(z_{\alpha_{\mathrm{p}}})$	$Q_{\mathrm{p}}(\widehat{z}_{1\cdot\mathrm{p}})$	z^*_{p}	$Q_{\mathrm{p}}(z_{\alpha_{\mathrm{p}}})$	$Q_{\mathrm{p}}(\widehat{z}_{1\cdot\mathrm{p}})$	z^*_{p}	$Q_{\mathrm{p}}(z_{\alpha_{\mathrm{p}}})$	$Q_{\mathrm{p}}(\widehat{z}_{1\cdot\mathrm{p}})$
20	20	20	30	2.414	0.920	0.952	2.440	0.918	0.952	2.599	0.905	0.952
			60	2.342	0.928	0.951	2.362	0.925	0.951	2.474	0.916	0.952
			90	2.295	0.932	0.952	2.320	0.929	0.952	2.400	0.922	0.953
			120	2.291	0.932	0.950	2.285	0.933	0.953	2.370	0.924	0.953
			150	2.259	0.936	0.952	2.264	0.935	0.953	2.353	0.926	0.952
			200	2.237	0.938	0.952	2.242	0.937	0.953	2.321	0.929	0.952
30	20	10	30	2.398	0.923	0.953	2.443	0.919	0.952	2.597	0.907	0.952
			60	2.319	0.930	0.953	2.339	0.928	0.954	2.456	0.918	0.953
			90	2.288	0.933	0.953	2.298	0.932	0.953	2.403	0.922	0.953
			120	2.259	0.936	0.953	2.290	0.932	0.952	2.354	0.926	0.954
			150	2.230	0.938	0.955	2.256	0.936	0.954	2.340	0.928	0.953
			200	2.233	0.939	0.953	2.234	0.938	0.954	2.311	0.930	0.953
40	10	10	30	2.393	0.923	0.954	2.441	0.918	0.952	2.599	0.908	0.952
			60	2.315	0.929	0.953	2.343	0.928	0.953	2.450	0.920	0.953
			90	2.278	0.934	0.954	2.284	0.932	0.954	2.399	0.923	0.953
			120	2.255	0.936	0.954	2.278	0.934	0.953	2.352	0.927	0.954
			150	2.253	0.937	0.953	2.256	0.936	0.954	2.325	0.929	0.954
			200	2.229	0.939	0.953	2.239	0.938	0.953	2.298	0.932	0.954

Table 2: The attained confidence level for pairwise comparisons when $k = 6$.

				Σ_1			Σ_2			Σ_3		
N_{12}	N_{34}	N_{56}	p	z^*_{p}	$Q_{\mathrm{p}}(z_{\alpha_{\mathrm{p}}})$	$Q_{\mathrm{p}}(\widehat{z}_{1\cdot\mathrm{p}})$	z^*_{p}	$Q_{\mathrm{p}}(z_{\alpha_{\mathrm{p}}})$	$Q_{\mathrm{p}}(\widehat{z}_{1\cdot\mathrm{p}})$	z^*_{p}	$Q_{\mathrm{p}}(z_{\alpha_{\mathrm{p}}})$	$Q_{\mathrm{p}}(\widehat{z}_{1\cdot\mathrm{p}})$
10	10	10	30	3.250	0.876	0.952	3.318	0.868	0.952	3.635	0.834	0.953
			60	3.093	0.897	0.953	3.145	0.890	0.952	3.384	0.861	0.951
			90	3.013	0.907	0.954	3.062	0.901	0.953	3.247	0.878	0.953
			120	2.961	0.915	0.955	3.013	0.908	0.953	3.177	0.885	0.953
			150	2.945	0.918	0.954	2.976	0.912	0.954	3.134	0.892	0.952
			200	2.909	0.922	0.954	2.950	0.917	0.952	3.062	0.901	0.954
15	10	5	30	3.224	0.882	0.955	3.303	0.871	0.954	3.609	0.838	0.955
			60	3.071	0.900	0.955	3.112	0.896	0.955	3.351	0.867	0.954
			90	2.997	0.911	0.956	3.038	0.905	0.955	3.227	0.881	0.955
			120	2.961	0.916	0.955	2.989	0.913	0.955	3.161	0.890	0.954
			150	2.935	0.919	0.955	2.953	0.916	0.956	3.111	0.897	0.954
			200	2.902	0.924	0.955	2.911	0.921	0.956	3.058	0.903	0.954
20	5	5	30	3.207	0.884	0.956	3.290	0.876	0.955	3.609	0.842	0.955
			60	3.053	0.905	0.956	3.110	0.896	0.956	3.337	0.871	0.956
			90	2.990	0.913	0.956	3.033	0.907	0.955	3.213	0.885	0.956
			120	2.945	0.918	0.956	2.997	0.913	0.955	3.150	0.893	0.955
			150	2.923	0.922	0.956	2.946	0.918	0.956	3.101	0.899	0.956
			200	2.891	0.927	0.956	2.905	0.925	0.957	3.044	0.906	0.956

Table 3: The attained confidence level for comparisons with a control when $k = 3$.

				Σ_1			Σ_2			Σ_3		
N_1	N_2	N_3	p	z_c^*	$Q_c(z_{\alpha_c})$	$Q_c(\widehat{z}_{1\cdot c})$	z_c^*	$Q_c(z_{\alpha_c})$	$Q_c(\widehat{z}_{1\cdot c})$	z_c^*	$Q_c(z_{\alpha_c})$	$Q_c(\widehat{z}_{1\cdot c})$
20	20	20	30	2.197	0.927	0.951	2.221	0.926	0.951	2.330	0.918	0.952
			60	2.123	0.933	0.952	2.152	0.931	0.951	2.241	0.924	0.951
			90	2.095	0.936	0.952	2.133	0.933	0.950	2.203	0.927	0.950
			120	2.094	0.936	0.950	2.091	0.936	0.952	2.148	0.931	0.952
			150	2.062	0.939	0.952	2.076	0.938	0.952	2.131	0.933	0.952
			200	2.056	0.940	0.951	2.065	0.938	0.952	2.112	0.935	0.952
30	20	10	30	2.223	0.924	0.949	2.243	0.923	0.949	2.367	0.914	0.949
			60	2.138	0.931	0.950	2.168	0.928	0.949	2.279	0.920	0.948
			90	2.110	0.934	0.950	2.132	0.932	0.950	2.202	0.926	0.950
			120	2.094	0.936	0.950	2.115	0.934	0.949	2.176	0.927	0.950
			150	2.079	0.937	0.950	2.096	0.935	0.950	2.166	0.929	0.949
			200	2.063	0.938	0.951	2.066	0.938	0.951	2.136	0.932	0.950
40	10	10	30	2.236	0.922	0.947	2.254	0.921	0.948	2.379	0.913	0.948
			60	2.146	0.931	0.950	2.149	0.930	0.951	2.253	0.921	0.950
			90	2.127	0.932	0.949	2.136	0.931	0.949	2.218	0.924	0.949
			120	2.090	0.936	0.950	2.119	0.934	0.949	2.183	0.927	0.949
			150	2.078	0.937	0.951	2.102	0.934	0.950	2.160	0.929	0.949
			200	2.070	0.938	0.950	2.082	0.937	0.950	2.122	0.933	0.951

Table 4: The attained confidence level for comparisons with a control when $k = 6$.

				Σ_1			Σ_2			Σ_3		
N_{12}	N_{34}	N_{56}	p	z_c^*	$Q_c(z_{\alpha_c})$	$Q_c(\widehat{z}_{1\cdot c})$	z_c^*	$Q_c(z_{\alpha_c})$	$Q_c(\widehat{z}_{1\cdot c})$	z_c^*	$Q_c(z_{\alpha_c})$	$Q_c(\widehat{z}_{1\cdot c})$
10	10	10	30	2.716	0.904	0.950	2.730	0.905	0.952	2.934	0.888	0.953
			60	2.565	0.922	0.954	2.620	0.918	0.952	2.768	0.902	0.952
			90	2.512	0.928	0.955	2.562	0.923	0.952	2.674	0.912	0.953
			120	2.499	0.929	0.953	2.524	0.927	0.953	2.636	0.916	0.953
			150	2.493	0.930	0.952	2.493	0.930	0.954	2.610	0.918	0.952
			200	2.452	0.935	0.954	2.468	0.933	0.954	2.555	0.923	0.954
15	10	5	30	2.716	0.904	0.950	2.759	0.899	0.950	2.957	0.882	0.951
			60	2.589	0.917	0.952	2.624	0.915	0.951	2.795	0.896	0.950
			90	2.554	0.922	0.951	2.584	0.919	0.951	2.712	0.905	0.950
			120	2.519	0.925	0.952	2.550	0.922	0.951	2.650	0.910	0.951
			150	2.503	0.928	0.951	2.517	0.925	0.952	2.622	0.914	0.951
			200	2.487	0.930	0.951	2.497	0.929	0.951	2.581	0.919	0.951
20	5	5	30	2.723	0.903	0.949	2.775	0.897	0.949	2.986	0.875	0.949
			60	2.613	0.915	0.950	2.638	0.910	0.950	2.808	0.892	0.949
			90	2.574	0.919	0.949	2.579	0.919	0.951	2.724	0.903	0.949
			120	2.529	0.924	0.950	2.537	0.923	0.952	2.669	0.908	0.950
			150	2.509	0.927	0.951	2.531	0.924	0.950	2.646	0.911	0.949
			200	2.484	0.930	0.951	2.506	0.928	0.950	2.591	0.917	0.950

Table 5: The robustness for pairwise comparisons.

N_1	N_2	N_3	p	D1	D2	D3	D4	D5
20	20	20	30	0.952	0.972	0.958	0.965	0.953
			60	0.951	0.982	0.960	0.965	0.952
			90	0.952	0.987	0.959	0.964	0.953
			120	0.950	0.991	0.959	0.965	0.952
			150	0.952	0.993	0.960	0.964	0.951
			200	0.952	0.996	0.961	0.965	0.952
30	20	10	30	0.953	0.959	0.955	0.956	0.953
			60	0.954	0.963	0.956	0.957	0.953
			90	0.953	0.965	0.955	0.957	0.952
			120	0.953	0.968	0.955	0.956	0.953
			150	0.954	0.969	0.955	0.957	0.954
			200	0.954	0.970	0.956	0.957	0.953
40	10	10	30	0.954	0.947	0.950	0.948	0.953
			60	0.953	0.944	0.950	0.949	0.954
			90	0.954	0.943	0.951	0.948	0.955
			120	0.954	0.942	0.951	0.949	0.953
			150	0.953	0.940	0.951	0.947	0.954
			200	0.953	0.940	0.950	0.948	0.954

Table 6: The robustness for comparisons with a control.

N_1	N_2	N_3	p	D1	D2	D3	D4	D5
20	20	20	30	0.951	0.969	0.958	0.961	0.949
			60	0.952	0.978	0.958	0.963	0.950
			90	0.952	0.984	0.957	0.962	0.951
			120	0.950	0.988	0.958	0.962	0.953
			150	0.952	0.991	0.957	0.962	0.951
			200	0.951	0.994	0.958	0.962	0.951
30	20	10	30	0.949	0.956	0.952	0.953	0.951
			60	0.950	0.958	0.953	0.955	0.950
			90	0.950	0.960	0.953	0.955	0.950
			120	0.950	0.962	0.951	0.953	0.950
			150	0.950	0.963	0.953	0.953	0.951
			200	0.951	0.964	0.953	0.955	0.951
40	10	10	30	0.947	0.938	0.945	0.941	0.950
			60	0.950	0.931	0.945	0.940	0.949
			90	0.949	0.927	0.944	0.941	0.951
			120	0.950	0.925	0.945	0.941	0.950
			150	0.951	0.923	0.946	0.943	0.949
			200	0.950	0.919	0.945	0.942	0.950

Table 7: The power for pairwise comparisons.

N_1	N_2	N_3	θ	$r = 0.25$	$r = 0.50$	$r = 0.75$	$r = 1.00$
20	20	20	0.1	0.002	0.004	0.006	0.010
			0.3	0.056	0.310	0.669	0.887
			0.5	0.594	0.975	0.983	0.982
			0.7	0.973	0.983	0.983	0.982
			0.9	0.982	0.983	0.983	0.983
30	20	10	0.1	0.001	0.002	0.003	0.006
			0.3	0.036	0.235	0.539	0.777
			0.5	0.472	0.940	0.982	0.983
			0.7	0.936	0.983	0.982	0.982
			0.9	0.983	0.983	0.983	0.982
40	10	10	0.1	0.001	0.001	0.002	0.003
			0.3	0.018	0.144	0.422	0.707
			0.5	0.352	0.929	0.981	0.983
			0.7	0.922	0.983	0.983	0.982
			0.9	0.982	0.983	0.983	0.983

Table 8: The power for comparisons with a control.

N_1	N_2	N_3	θ	$r = 0.25$	$r = 0.50$	$r = 0.75$	$r = 1.00$
20	20	20	0.1	0.005	0.007	0.012	0.017
			0.3	0.084	0.387	0.740	0.927
			0.5	0.674	0.995	1.000	1.000
			0.7	0.994	1.000	1.000	1.000
			0.9	1.000	1.000	1.000	1.000
30	20	10	0.1	0.002	0.004	0.007	0.010
			0.3	0.058	0.305	0.620	0.837
			0.5	0.551	0.969	0.999	1.000
			0.7	0.966	1.000	1.000	1.000
			0.9	1.000	1.000	1.000	1.000
40	10	10	0.1	0.002	0.002	0.004	0.007
			0.3	0.033	0.202	0.507	0.778
			0.5	0.434	0.962	0.999	1.000
			0.7	0.955	1.000	1.000	1.000
			0.9	1.000	1.000	1.000	1.000