Optimization using Cross-Validation for Penalized Nonlinear Canonical Correlation Analysis

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Abstract

Hotelling (1936) proposed canonical correlation analysis (CCA) as a way to find the linear relationship between a pair of random vectors. However, CCA can not find nonlinear relationships between them since the method maximizes the correlation between linear combinations of the vectors. In order to find a nonlinear relationship, we convert the vectors through some conversion functions like a kernel functions. Then we find the nonlinear relationship in the original vectors through the conversion functions. However, this method which is used some conversion functions has a critical issue in that the maximized correlation occasionally becomes 1 even if there is no relationship between the random vectors (Hardoon *et al.*, 2004). Akaho (2000) proposed a penalized method that avoids this issue when the kernel functions are used for conversion. In this method, however, methods have not been proposed for optimizing the penalty and parameters, even though the results heavily depend on these parameters. In this paper, we propose an optimization method for the penalty and other parameters, based on the cross-validation method.

Key words: Canonical Correlation Analysis; Cross-Validation; Nonlinear Relationship; Penalized Method.

1. Introduction

Let $\boldsymbol{y}$ and $\boldsymbol{x}$ be q_0- and p_0-dimensional random vectors with $E[\boldsymbol{y}] = \mathbf{0}_{q_0}$ and $E[\boldsymbol{x}] = \mathbf{0}_{p_0}$, where $\mathbf{0}_\ell$ is an ℓ-dimensional vector of zeros. As a method for determining if there is a linear relationship between $\boldsymbol{y}$ and $\boldsymbol{x}$, Hotelling (1936) proposed canonical correlation analysis (CCA). This method is formulated as follows:

$$\max_{\boldsymbol{a}\in\mathbb{R}^{q_0},\boldsymbol{b}\in\mathbb{R}^{p_0}} \boldsymbol{a}'\boldsymbol{\Sigma}_{yx}\boldsymbol{b} \text{ s.t. } \boldsymbol{a}'\boldsymbol{\Sigma}_{yy}\boldsymbol{a} = 1 \text{ and } \boldsymbol{b}'\boldsymbol{\Sigma}_{xx}\boldsymbol{b} = 1, \tag{1.1}$$

where $\boldsymbol{\Sigma}_{yx} = \mathrm{Cov}(\boldsymbol{y}, \boldsymbol{x})$, $\boldsymbol{\Sigma}_{yy} = \mathrm{Var}(\boldsymbol{y})$, and $\boldsymbol{\Sigma}_{xx} = \mathrm{Var}(\boldsymbol{x})$; and we assume $\det(\boldsymbol{\Sigma}_{yy}) \neq 0$ and $\det(\boldsymbol{\Sigma}_{xx}) \neq 0$. Using the Lagrange method of undetermined multipliers, the solutions of $\boldsymbol{a}$ and $\boldsymbol{b}$ are equivalent to the eigenvectors of $\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{yx}'$ and $\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{yx}'\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}$, respectively. More details of CCA can be found in Muirhead (1982), Gittins (1985), Srivastava (2002), and Weenink (2003). This

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method is currently being used for data analysis (see, e.g., Doeswijk, *et al.*, 2011). CCA, however, can not find nonlinear relationships between $\boldsymbol{y}$ and $\boldsymbol{x}$, since the maximization term in (1.1) is equivalent to $\mathrm{Cov}(\boldsymbol{a}'\boldsymbol{y}, \boldsymbol{b}'\boldsymbol{x})$, which evaluates the linear relationship between linear combinations $\boldsymbol{a}'\boldsymbol{y}$ and $\boldsymbol{b}'\boldsymbol{x}$. A similar problem also occurs in the ordinary linear regression model.

In order to find a nonlinear relationship between $\boldsymbol{y}$ and $\boldsymbol{x}$, we consider converting them by using some functions like a kernel functions. CCA can then find a nonlinear relationship between the converted $\boldsymbol{y}$ and $\boldsymbol{x}$. This method is referred to as nonlinear canonical correlation analysis (NCCA), and it is discussed in Section 2. Hardoon, *et al.* (2004) pointed out that NCCA has a critical issue which is shown in Section 2.

Using the same idea as is used in the penalized nonlinear regression model, Akaho (2000) proposed a penalized NCCA when the kernel functions are used for the conversion functions. We will refer to the penalized NCCA as PNCCA even when it uses any conversion functions instead of the kernel functions. In PNCCA, no criteria have yet been developed for for optimizing the penalty and other parameters. Because of this, it is difficult to know how to evaluate the result of PNCCA. In particular, determining how to optimize the penalty and other parameters is important, since the result of PNCCA depends heavily on these parameters.

We propose a method based on cross-validation (CV), to optimize the penalty and other parameters in the PNCCA. Details of the proposed optimization method are presented in Section 3.

The remainder of the present paper is organized as follows: In Section 2, we present more details of CCA, NCCA, and PNCCA. In Section 3, we propose the CV method for optimizing several parameters in PNCCA. In Section 4, we use numerical studies to compare CCA, NCCA, and PNCCA based on the optimized parameters. In Section 5, we present our conclusions. Using this CV method, we can select the variables in $\boldsymbol{y}$ and $\boldsymbol{x}$; this is illustrated in the Appendix.

2. CCA, NCCA, and PNCCA

In this section, we discuss CCA, NCCA, and PNCCA. We first discuss CCA, which is expressed as (1.1). The Lagrange method of undetermined multipliers $\mathcal{L}(\theta_a, \theta_b, \boldsymbol{a}, \boldsymbol{b})$, where θ_a and θ_b are nonnegative undetermined constant terms, is then usually applied to (1.1). We solve the resulting simultaneous equations: $\partial\mathcal{L}(\theta_a, \theta_b, \boldsymbol{a}, \boldsymbol{b})/(\partial\boldsymbol{a})|_{\boldsymbol{a}=\tilde{\boldsymbol{a}}} = \boldsymbol{0}_{q_0}$, $\partial\mathcal{L}(\theta_a, \theta_b, \boldsymbol{a}, \boldsymbol{b})/(\partial\boldsymbol{b})|_{\boldsymbol{b}=\tilde{\boldsymbol{b}}} = \boldsymbol{0}_{p_0}$, $\partial\mathcal{L}(\theta_a, \theta_b, \boldsymbol{a}, \boldsymbol{b})/(\partial\theta_a)|_{\theta_a=\tilde{\theta_a}} = 0$, and $\partial\mathcal{L}(\theta_a, \theta_b, \boldsymbol{a}, \boldsymbol{b})/(\partial\theta_b)|_{\theta_b=\tilde{\theta_b}} = 0$. Thus, CCA is the same as solving the following eigenvalue problem:

$$\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{yx}'\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}\tilde{\boldsymbol{b}} = \tilde{\theta}^2\tilde{\boldsymbol{b}}, \tag{2.1}$$

and $\tilde{\boldsymbol{a}} = \boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}\tilde{\boldsymbol{b}}/\tilde{\theta}$, where $\tilde{\theta} = \tilde{\boldsymbol{a}}'\boldsymbol{\Sigma}_{yx}\tilde{\boldsymbol{b}} > 0$; note that $\tilde{\theta} = \tilde{\theta}_a = \tilde{\theta}_b$. Hence, solving the eigenvalue problem in (2.1) and using the largest eigenvalue and the corresponding eigenvector, we can solve the maximization problem in (1.1) under several conditions. More details of CCA can be found in e.g., Muirhead (1982).

Since CCA can not find a nonlinear relationship between $\boldsymbol{y}$ and $\boldsymbol{x}$, we consider using NCCA on converted $\boldsymbol{y}$ and $\boldsymbol{x}$. Let $\boldsymbol{z} = \boldsymbol{\phi}(\boldsymbol{y})$ and $\boldsymbol{w} = \boldsymbol{\varphi}(\boldsymbol{x})$, where $\boldsymbol{\phi}: \mathbb{R}^{q_0} \to \mathbb{R}^{q_1}$ and $\boldsymbol{\varphi}: \mathbb{R}^{p_0} \to \mathbb{R}^{p_1}$, and suppose that $E[\boldsymbol{z}] = \boldsymbol{0}_{q_1}$ and $E[\boldsymbol{w}] = \boldsymbol{0}_{p_1}$. The nonlinear relationship between $\boldsymbol{y}$ and $\boldsymbol{x}$ can be found through the conversion functions $\boldsymbol{\phi}(\cdot)$ and $\boldsymbol{\varphi}(\cdot)$ by using CCA on $\boldsymbol{z}$ and $\boldsymbol{w}$ instead of on $\boldsymbol{y}$ and $\boldsymbol{x}$. However, Hardoon, *et al.* (2004) pointed out that, even if there is no relationship between $\boldsymbol{y}$ and $\boldsymbol{x}$, in some situations, NCCA can encounter the following problem:

$$1 = \max_{\boldsymbol{c}\in\mathbb{R}^{q_1}, \boldsymbol{d}\in\mathbb{R}^{p_1}} \boldsymbol{c}'\boldsymbol{\Sigma}_{zw}\boldsymbol{d} \text{ s.t. } \boldsymbol{c}'\boldsymbol{\Sigma}_{zz}\boldsymbol{c} = 1 \text{ and } \boldsymbol{d}'\boldsymbol{\Sigma}_{ww}\boldsymbol{d} = 1, \tag{2.2}$$

where $\boldsymbol{\Sigma}_{zw} = \mathrm{Cov}(\boldsymbol{z}, \boldsymbol{w})$, $\boldsymbol{\Sigma}_{zz} = \mathrm{Var}(\boldsymbol{z})$, and $\boldsymbol{\Sigma}_{ww} = \mathrm{Var}(\boldsymbol{w})$.

In order to avoid this problem, Akaho (2000) proposed PNCCA when the kernel functions are used for conversion functions. This is the primary method we consider in this paper. As with an ordinary penalized estimator in the nonlinear regression model, the penalty term in PNCCA is set to shrink the norms of $\boldsymbol{c}$ and $\boldsymbol{d}$. If there exists $\boldsymbol{\phi}^{-1}$ such that $\boldsymbol{y} = \boldsymbol{\phi}^{-1}(\boldsymbol{z})$, then, since we can obtain $\boldsymbol{w}_2 = \boldsymbol{\phi}^{-1}(\boldsymbol{\varphi}(\boldsymbol{x}))$, we can use $\boldsymbol{y}$ and $\boldsymbol{w}_2$ instead of $\boldsymbol{z}$ and $\boldsymbol{w}$. Thus, in order to simplify, we consider using the PNCCA with $\boldsymbol{y}$ and $\boldsymbol{w} = \boldsymbol{\varphi}(\boldsymbol{x})$, and we assume $E[\boldsymbol{w}] = \boldsymbol{0}_{p_1}$ and $\det(\boldsymbol{\Sigma}_{ww}) \neq 0$. Since, in our setting, only $\boldsymbol{x}$ is converted, the penalty term is set to shrink the norm of $\boldsymbol{d}$. PNCCA is then expressed as follows:

$$\max_{\boldsymbol{a}\in\mathbb{R}^{q_0}, \boldsymbol{d}\in\mathbb{R}^{p_1}} \boldsymbol{a}'\boldsymbol{\Sigma}_{yw}\boldsymbol{d} \text{ s.t. } \boldsymbol{a}'\boldsymbol{\Sigma}_{yy}\boldsymbol{a} = 1 \text{ and } \boldsymbol{d}'(\boldsymbol{\Sigma}_{ww} + \lambda\boldsymbol{P})\boldsymbol{d} = 1, \tag{2.3}$$

where $\boldsymbol{\Sigma}_{yw} = \mathrm{Cov}(\boldsymbol{y}, \boldsymbol{w})$, λ is a nonnegative penalty parameter, and $\boldsymbol{P}$ is a known $p_1 \times p_1$ nonnegative definite matrix. Note that $\lambda\boldsymbol{d}'\boldsymbol{P}\boldsymbol{d}$ is the penalty term in (2.3), and it is always nonnegative for any $\boldsymbol{d} \in \mathbb{R}^{p_1}$. The same as for CCA in (1.1), in order to solve the maximization problem under various conditions in (2.3), we use the Lagrange method of undetermined multipliers:

$$\mathcal{L}_P(\eta_a, \eta_d, \boldsymbol{a}, \boldsymbol{d}, \lambda|\boldsymbol{P}) = \boldsymbol{a}'\boldsymbol{\Sigma}_{yw}\boldsymbol{d} - \frac{\eta_a}{2}(\boldsymbol{a}'\boldsymbol{\Sigma}_{yy}\boldsymbol{a} - 1) - \frac{\eta_d}{2}\left\{\boldsymbol{d}'(\boldsymbol{\Sigma}_{ww} + \lambda\boldsymbol{P})\boldsymbol{d} - 1\right\},$$

where η_a and η_d are undetermined nonnegative constants. For fixed λ, solving the simultaneous equations $\partial\mathcal{L}_P(\eta_a, \eta_d, \boldsymbol{a}, \boldsymbol{d}, \lambda|\boldsymbol{P})/(\partial\boldsymbol{a})|_{\boldsymbol{a}=\tilde{\boldsymbol{a}}_\lambda} = \boldsymbol{0}_{q_0}$, $\partial\mathcal{L}_P(\eta_a, \eta_d, \boldsymbol{a}, \boldsymbol{d}, \lambda|\boldsymbol{P})/(\partial\boldsymbol{d})|_{\boldsymbol{d}=\tilde{\boldsymbol{d}}_\lambda} = \boldsymbol{0}_{p_1}$, $\partial\mathcal{L}_P(\eta_a, \eta_d, \boldsymbol{a}, \boldsymbol{d}, \lambda|\boldsymbol{P})/(\partial\eta_a)|_{\eta_a=\tilde{\eta}_{a,\lambda}} = 0$, and $\partial\mathcal{L}_P(\eta_a, \eta_d, \boldsymbol{a}, \boldsymbol{d}, \lambda|\boldsymbol{P})/(\partial\eta_d)|_{\eta_d=\tilde{\eta}_{d,\lambda}} = 0$, coincides with the following eigenvalue problem:

$$(\boldsymbol{\Sigma}_{ww} + \lambda\boldsymbol{P})^{-1}\boldsymbol{\Sigma}_{yw}'\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}\tilde{\boldsymbol{d}}_\lambda = \tilde{\eta}_\lambda^2\tilde{\boldsymbol{d}}_\lambda,$$

and $\tilde{\boldsymbol{a}}_\lambda = \boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yw}\tilde{\boldsymbol{d}}_\lambda/\tilde{\eta}_\lambda$, where $\tilde{\eta}_\lambda = \tilde{\boldsymbol{a}}_\lambda'\boldsymbol{\Sigma}_{yw}\tilde{\boldsymbol{d}}_\lambda > 0$; note that $\tilde{\eta}_\lambda = \tilde{\eta}_{a,\lambda} = \tilde{\eta}_{d,\lambda}$. Thus, when the penalty parameter λ is given, the largest eigenvalue and the corresponding eigenvector of the above eigenvalue problem solve (2.3).

However, although it is important, there are no known optimization methods for λ. In the next section, we propose a CV method for optimizing λ and some of the parameters in the conversion function $\boldsymbol{\varphi}(\cdot)$.

3. Proposed Method

In this section, we propose a CV method for optimizing the penalty and other parameters in PNCCA. Since $\boldsymbol{\Sigma}_{ww}$, $\boldsymbol{\Sigma}_{yw}$, and $\boldsymbol{\Sigma}_{yy}$ are unknown matrices, we use their unbiased estimators to estimate $\tilde{\eta}_\lambda$, $\tilde{\boldsymbol{a}}_\lambda$, and $\tilde{\boldsymbol{d}}_\lambda$. Let $\boldsymbol{S}_{ww}$, $\boldsymbol{S}_{yw}$, and $\boldsymbol{S}_{yy}$ be the unbiased estimators for $\boldsymbol{\Sigma}_{ww}$, $\boldsymbol{\Sigma}_{yw}$, and $\boldsymbol{\Sigma}_{yy}$, respectively, based on the sample $\{\boldsymbol{y}_i, \boldsymbol{x}_i\}_{i=1,\ldots,n}$ and $\boldsymbol{w}_i = \boldsymbol{\varphi}(\boldsymbol{x}_i)$. Let $\hat{\eta}_\lambda$ (> 0), $\hat{\boldsymbol{a}}_\lambda$, and $\hat{\boldsymbol{d}}_\lambda$ be the estimators of $\tilde{\eta}_\lambda$, $\tilde{\boldsymbol{a}}_\lambda$, and $\tilde{\boldsymbol{d}}_\lambda$, respectively. Then $\hat{\eta}_\lambda^2$ and $\hat{\boldsymbol{d}}_\lambda$ are derived as the largest eigenvalue and the corresponding eigenvector of $(\boldsymbol{S}_{ww} + \lambda \boldsymbol{P})^{-1}\boldsymbol{S}'_{yw}\boldsymbol{S}_{yy}^{-1}\boldsymbol{S}_{yw}$, and $\hat{\boldsymbol{a}}_\lambda = \boldsymbol{S}_{yy}^{-1}\boldsymbol{S}_{yw}\hat{\boldsymbol{d}}_\lambda/\hat{\eta}_\lambda$.

We consider creating an objective function in order to propose a criterion for optimizing several of the parameters in the PNCCA. Recall that $\hat{\boldsymbol{a}}_\lambda$ and $\hat{\boldsymbol{d}}_\lambda$ are derived from $\{\boldsymbol{y}_i.\boldsymbol{x}_i\}_{i=1,\ldots,n}$. Thus, in order to evaluate $\hat{\boldsymbol{a}}_\lambda$ and $\hat{\boldsymbol{d}}_\lambda$, we consider the following evaluation function:

$$R^* = E[\hat{\boldsymbol{a}}'_\lambda \boldsymbol{\Sigma}_{yw}\hat{\boldsymbol{d}}_\lambda]. \tag{3.1}$$

Maximizing the above function, we can optimize the parameters in the PNCCA. However, $\boldsymbol{\Sigma}_{yw}$ is an unknown covariance matrix. We therefore consider using an estimator of $\boldsymbol{\Sigma}_{yw}$ that does not depend on $\{\boldsymbol{y}_i, \boldsymbol{x}_i\}_{i=1,\ldots,n}$ in order to estimate R^* in (3.1).

Let $\boldsymbol{y}^*$ and $\boldsymbol{x}^*$ be new variables that are obtained independently with $\{\boldsymbol{y}_i, \boldsymbol{x}_i\}_{i=1,\ldots,n}$, and let $\boldsymbol{S}_{y^*w^*}$ be the covariance matrix between $\boldsymbol{y}^*$ and $\boldsymbol{w}^* = \boldsymbol{\varphi}(\boldsymbol{x}^*)$. Then $\boldsymbol{S}_{y^*w^*}$ is an estimator of $\boldsymbol{\Sigma}_{yw}$. Based on $\boldsymbol{S}_{y^*w^*}$, the evaluation function R^* in (3.1) is estimated by using the average of the following value:

$$\hat{R}^* = \hat{\boldsymbol{a}}'_\lambda \boldsymbol{S}_{y^*w^*}\hat{\boldsymbol{d}}_\lambda. \tag{3.2}$$

Nevertheless, these evaluation functions R^* and $\hat{R}^*$ in (3.1) and (3.2) can not be used directly for optimizing the parameters in the PNCCA since $\boldsymbol{y}^*$ and $\boldsymbol{w}^*$ are not obtained. We thus use the CV method to optimize several parameters in the PNCCA.

Let $\boldsymbol{V} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)'$, where $\boldsymbol{v}_i = (\boldsymbol{y}'_i, \boldsymbol{w}'_i)'$, $(i = 1, \ldots, n)$ and $\boldsymbol{w}_i = \boldsymbol{\varphi}(\boldsymbol{x}_i)$. The essence of the proposed method is to obtain a matrix that is an alternative to $\boldsymbol{S}_{y^*w^*}$. The ordinary CV method, which is used for selecting variables in the linear regression model, is based on $\boldsymbol{V}^{[-i]}$, which is obtained by deleting $\boldsymbol{v}'_i$ from $\boldsymbol{V}$ for each i. When we use the ordinary CV method, we can not obtain an evaluation function to replace $\hat{R}^*$ in (3.2). This is why, in the ordinary CV method, $\hat{\boldsymbol{a}}_\lambda$ and $\hat{\boldsymbol{d}}_\lambda$ are derived from $\boldsymbol{V}^{[-i]}$. However, an alternative to $\boldsymbol{S}_{y^*w^*}$ can not be obtained since it can not obtain sufficient information from only one sample $\boldsymbol{v}_i$. Hence the ordinary CV method can not be used to replace $\hat{R}^*$.

We now use $\boldsymbol{v}_i = (\boldsymbol{y}_i', \boldsymbol{w}_i')'$ and $\boldsymbol{v}_j = (\boldsymbol{y}_j', \boldsymbol{w}_j')'$, $(i \neq j)$ to derive an alternative to $\boldsymbol{S}_{y^*w^*}$, which can be defined as

$$\hat{\boldsymbol{S}}_{[i,j]} = \frac{(\boldsymbol{y}_i - \boldsymbol{y}_j)(\boldsymbol{w}_j - \boldsymbol{w}_i)'}{4}, \ (i = 1, \ldots, n; \ j = 1, \ldots, n; \ i \neq j),$$

since $(\boldsymbol{y}_i + \boldsymbol{y}_j)/2$ and $(\boldsymbol{w}_i + \boldsymbol{w}_j)/2$ are the sample means based on $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$, $(i \neq j)$, and the sample covariance matrix between $\boldsymbol{y}_i$ and $\boldsymbol{w}_j$ is derived as $(\boldsymbol{y}_i - (\boldsymbol{y}_i + \boldsymbol{y}_j)/2)(\boldsymbol{w}_j - (\boldsymbol{w}_i + \boldsymbol{w}_j)/2)'$. Note that $\hat{\boldsymbol{S}}_{[i,j]} = \hat{\boldsymbol{S}}_{[j,i]}$ for any i and j, $i \neq j$. Let $\boldsymbol{V}^{[-i,-j]}$, $(i = 1, \ldots, n; \ j = 1, \ldots, n; \ i \neq j)$ be obtained by deleting $\boldsymbol{v}_i'$ and $\boldsymbol{v}_j'$, $(i \neq j)$ from $\boldsymbol{V}$. Furthermore, let $\boldsymbol{S}_{ww}^{[-i,-j]}$, $\boldsymbol{S}_{yw}^{[-i,-j]}$, and $\boldsymbol{S}_{yy}^{[-i,-j]}$ be derived by using $\boldsymbol{V}^{[-i,-j]}$ and be based on the ordinary estimation method for covariance matrices. Then, if λ is given, $\hat{\boldsymbol{d}}_\lambda^{[-i,-j]}$ is derived as the eigenvector that corresponds to the largest eigenvalue of $(\boldsymbol{S}_{ww}^{[-i,-j]} + \lambda \boldsymbol{P})^{-1} \boldsymbol{S}_{yw}^{[-i,-j]\prime} \boldsymbol{S}_{yy}^{[-i,-j]^{-1}} \boldsymbol{S}_{yw}^{[-i,-j]}$. Using $\hat{\boldsymbol{d}}_\lambda^{[-i,-j]}$ and the largest eigenvalue $(\hat{\theta}_\lambda^{[-i,-j]})^2$, $\hat{\boldsymbol{a}}_\lambda^{[-i,-j]}$ is obtained as $\hat{\boldsymbol{a}}_\lambda^{[-i,-j]} = \boldsymbol{S}_{yy}^{[-i,-j]^{-1}} \boldsymbol{S}_{yw}^{[-i,-j]} \hat{\boldsymbol{d}}_\lambda^{[-i,-j]} / \hat{\theta}_\lambda^{[-i,-j]}$, where $\hat{\theta}_\lambda^{[-i,-j]} > 0$. Note that $\hat{\boldsymbol{a}}_\lambda^{[-i,-j]}$ and $\hat{\boldsymbol{d}}_\lambda^{[-i,-j]}$ are derived from $\boldsymbol{V}^{[-i,-j]}$, and $\hat{\boldsymbol{S}}_{[i,j]}$ is derived from $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$, $(i \neq j)$, which are not used for $\hat{\boldsymbol{a}}_\lambda^{[-i,-j]}$ and $\hat{\boldsymbol{d}}_\lambda^{[-i,-j]}$. In order to optimize the penalty parameter λ and the other parameters, $\hat{\boldsymbol{a}}_\lambda^{[-i,-j]}$ and $\hat{\boldsymbol{d}}_\lambda^{[-i,-j]}$ for each i and j, we evaluate $T = \sum_{i \neq j} |c_{ij}|$ where

$$c_{ij} = \hat{\boldsymbol{a}}_\lambda^{[-i,-j]\prime} \hat{\boldsymbol{S}}_{[i,j]} \hat{\boldsymbol{d}}_\lambda^{[-i,-j]}, \ (i = 1, \ldots, n; \ j = 1, \ldots, n; \ i \neq j). \tag{3.3}$$

Thus, for example, the penalty parameter λ in the PNCCA can be optimized as $\hat{\lambda} = \arg\max_{\lambda \geq 0} T$.

Let α and β be the independent parameters in the PNCCA other than the penalty parameter λ. Several parameters are optimized in the following algorithm:

1. Given α and β, we optimize $\hat{\lambda}$, which is regarded as $\hat{\lambda}(\alpha, \beta)$.

2. Given β, we obtain $\hat{\lambda}(\hat{\alpha}, \beta)$ by maximizing $\hat{\boldsymbol{a}}_{\hat{\lambda}(\alpha,\beta)}' \boldsymbol{S}_{yw} \hat{\boldsymbol{b}}_{\hat{\lambda}(\alpha,\beta)}$ by changing α.

3. We obtain $\hat{\lambda}(\hat{\alpha}, \hat{\beta})$ by maximizing $\hat{\boldsymbol{a}}_{\hat{\lambda}(\hat{\alpha},\beta)}' \boldsymbol{S}_{yw} \hat{\boldsymbol{b}}_{\hat{\lambda}(\hat{\alpha},\beta)}$ by changing β.

If we can use some other optimization method for two or more parameters, we can combine steps 2 and 3. Note that we can repeat this algorithm when we need to optimize more than two parameters.

4. Numerical Study

In this section, we use numerical simulations to compare CCA, NCCA, and PNCCA optimized with the proposed CV method. Note that NCCA can be defined by the same form as PNCCA in (2.3) and fixed $\lambda = 0$. Let $\boldsymbol{\Delta}_r(\rho)$ be an $r \times r$ matrix whose (i,j)th element is derived as $\rho^{|i-j|}$. The $n \times p_0$ matrix $\boldsymbol{X}$ is generated from $\boldsymbol{X} = \boldsymbol{U} \boldsymbol{\Delta}_{p_0}(\rho_x)^{1/2}$, where $\boldsymbol{U}$ is an $n \times p_0$ matrix whose elements were generated independently from the standard normal distribution. Then, $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ are derived as follows:

$$\begin{aligned}
&\text{(A)} \quad \boldsymbol{y}_i &&= \delta \boldsymbol{x}_i' \boldsymbol{x}_i \mathbf{1}_{q_0} + \boldsymbol{\varepsilon}_{i \cdot q_0}, \\
&\text{(B)} \quad \boldsymbol{y}_i &&= \delta (\boldsymbol{x}_i' \boldsymbol{x}_i / \max(x_{ij}^2), \sin(2\boldsymbol{x}_i' \boldsymbol{x}_i), \cos(2\boldsymbol{x}_i' \boldsymbol{x}_i))' + \boldsymbol{\varepsilon}_{i \cdot 3}, \\
&\text{(C)} \quad \boldsymbol{y}_i &&= \delta (\boldsymbol{x}_i' \boldsymbol{x}_i / \max(x_{ij}^2), \sin(2\boldsymbol{x}_i' \boldsymbol{x}_i), \cos(2\boldsymbol{x}_i' \boldsymbol{x}_i), \exp(-\boldsymbol{x}_i' \boldsymbol{x}_i / 4))' + \boldsymbol{\varepsilon}_{i \cdot 4},
\end{aligned}$$

where $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip_0})'$ is the ith row of the standardized $\boldsymbol{X}$, and $\boldsymbol{\varepsilon}_{i \cdot j}$ is generated independently from $N_j(\mathbf{0}_j, \boldsymbol{\Delta}_j(0.5))$, which is a j-dimensional multivariate normal distribution with mean $\mathbf{0}_j$ and covariance matrix $\boldsymbol{\Delta}_j(0.5)$.

Both the NCCA and PNCCA methods use an $n \times p_0$ matrix $\boldsymbol{W}$, whose (i,j)th element w_{ij} is converted as $w_{ij} = \exp\{-x_{ij}^2/(2h)\}$, and then standardized. We choose h by comparing the maximized correlation for each value $\{0.05, 0.1, 0.5, 1, 2, 5\}$ in each repetition. In PNCCA, the nonnegative penalty matrix $\boldsymbol{P}$ is set to $\boldsymbol{P} = \boldsymbol{K}'\boldsymbol{K}$, where $\boldsymbol{K} = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{p_0-2})'$ is a $(p_0 - 2) \times p_0$ matrix and $\boldsymbol{k}_j = (\mathbf{0}_{j-1}', 1, -2, 1, \mathbf{0}_{p_0-j-2}')', (j = 1, \ldots, p_0 - 2)$. (More details of $\boldsymbol{K}$ can be found in Green and Silverman (1994).) Since the 'arg max' operator is equivalent to the 'arg min' operator with the sign reversed, we select λ by using the Matlab 'fminbnd' function which is the Matlab 'fminsearch' in a specified region, and we restrict the region to 1 to exp(20) in order to shorten the computation time. Furthermore, in order to reduce computational tasks, we calculate c_{ij} in (3.3) for $i = 1, \ldots, n-1$ and $j = i+1$ for use in the CV method.

In order to derive R^* in (3.1), we set $n = 10,000$ and generate $\boldsymbol{X}$ for each p_0 and ρ_x, and standardize them. Then, from each transformation function and each parameter δ and q_0, we obtain $\boldsymbol{Y}$, which is also standardized. Note that $q_0 = 3$ when the transformation function is in (B), and $q_0 = 4$ when the transformation function is in (C). In CCA, $\boldsymbol{\Sigma}_{y^*y^*}$, $\boldsymbol{\Sigma}_{y^*x^*}$, and $\boldsymbol{\Sigma}_{x^*x^*}$ can be derived as the sample variance matrix of the standardized $\boldsymbol{Y}$, the sample covariance matrix of the standardized $\boldsymbol{Y}$ and $\boldsymbol{X}$, and the sample variance matrix of the standardized $\boldsymbol{X}$, respectively. In the NCCA and PNCCA methods, we convert $\boldsymbol{X}$ as above for each h and standardize the converted values. Then, $\boldsymbol{\Sigma}_{y^*w^*}$ and $\boldsymbol{\Sigma}_{w^*w^*}$ can be derived as the sample covariance matrix of standardized $\boldsymbol{Y}$ and $\boldsymbol{W}$ and the sample variance matrix of the standardized $\boldsymbol{W}$, respectively. Using these matrices, we evaluated the results of each method.

In order to evaluate the methods, we $\boldsymbol{X}$ and generated $\boldsymbol{Y}$ for $1,000$ repetitions. We used the standardized $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{W}$ in each repetition.

For the CCA method, for each repetition, we obtained the estimator of the covariance matrix $\boldsymbol{\Sigma}_{yx}$, that of the variance matrices $\boldsymbol{\Sigma}_{yy}$ and $\boldsymbol{\Sigma}_{xx}$ as $\boldsymbol{S}_{yx}$, $\boldsymbol{S}_{yy}$, and $\boldsymbol{S}_{xx}$, respectively. On the other hand, for the NCCA and PNCC methods, for each repetition, we obtained the estimator of the covariance matrix $\boldsymbol{\Sigma}_{yw}$, that of the variance matrices $\boldsymbol{\Sigma}_{yy}$ and $\boldsymbol{\Sigma}_{ww}$ as $\boldsymbol{S}_{yw}$, $\boldsymbol{S}_{yy}$, and $\boldsymbol{S}_{ww}$, respectively.

For each repetition with the CCA method (1.1), we calculated the maximized correlation under certain conditions by using $\boldsymbol{S}_{yx}$, $\boldsymbol{S}_{yy}$, and $\boldsymbol{S}_{xx}$ instead of $\boldsymbol{\Sigma}_{yx}$, $\boldsymbol{\Sigma}_{yy}$, and $\boldsymbol{\Sigma}_{xx}$, respectively. We denote the maximized correlation as $\hat{\theta}^2$, the eigenvector that corresponds to the largest eigenvalue of $\boldsymbol{S}_{xx}^{-1}\boldsymbol{S}_{yx}'\boldsymbol{S}_{xx}^{-1}\boldsymbol{S}_{xy}$, as $\hat{\boldsymbol{b}}_{\mathrm{C}}$, and then $\hat{\boldsymbol{a}}_{\mathrm{C}} = \boldsymbol{S}_{yy}^{-1}\boldsymbol{S}_{yx}\hat{\boldsymbol{b}}_{\mathrm{C}}/\hat{\theta}$ is derived where $\hat{\theta} > 0$. For each repetition with the NCCA method which can be defined as (2.3) with $\lambda = 0$, we calculated the maximized correlation under certain conditions and the optimized h, for which we used $\boldsymbol{S}_{yw}$, $\boldsymbol{S}_{yy}$, and $\boldsymbol{S}_{ww}$ instead of $\boldsymbol{\Sigma}_{yw}$, $\boldsymbol{\Sigma}_{yy}$, and $\boldsymbol{\Sigma}_{ww}$, respectively. We denote the maximized correlation as $\hat{\eta}_0^2$, the eigenvector that corresponds to the largest eigenvalue of $\boldsymbol{S}_{ww}^{-1}\boldsymbol{S}_{yw}'\boldsymbol{S}_{ww}^{-1}\boldsymbol{S}_{wy}$ as $\hat{\boldsymbol{d}}_{\mathrm{N}}$, and then $\hat{\boldsymbol{a}}_{\mathrm{N}} = \boldsymbol{S}_{yy}^{-1}\boldsymbol{S}_{yx}\hat{\boldsymbol{d}}_{\mathrm{N}}/\hat{\eta}_0$ is derived where $\hat{\eta}_0 > 0$. For each repetition with the PNCCA method (2.3), we calculated the maximized correlation under certain conditions by using the optimized λ and optimized h, and we used $\boldsymbol{S}_{yw}$, $\boldsymbol{S}_{yy}$, and $\boldsymbol{S}_{ww}$ instead of $\boldsymbol{\Sigma}_{yw}$, $\boldsymbol{\Sigma}_{yy}$, and $\boldsymbol{\Sigma}_{ww}$, respectively. We denote the maximized correlation as $\hat{\eta}_{\hat{\lambda}}^2$ and the eigenvector that corresponds to the largest eigenvalue of $(\boldsymbol{S}_{ww} + \hat{\lambda}\boldsymbol{P})^{-1}\boldsymbol{S}_{yw}'\boldsymbol{S}_{ww}^{-1}\boldsymbol{S}_{wy}$ as $\hat{\boldsymbol{d}}_{\mathrm{P}}$, where $\hat{\lambda}$ is the optimized penalty parameter based on the proposed CV method, and then $\hat{\boldsymbol{a}}_{\mathrm{P}} = \boldsymbol{S}_{yy}^{-1}\boldsymbol{S}_{yw}\hat{\boldsymbol{d}}_{\mathrm{P}}/\hat{\eta}_{\hat{\lambda}}$ is derived where $\hat{\eta}_{\hat{\lambda}} > 0$.

Firstly, we compared the CCA, NCCA, and PNCCA methods by using the average values of $\hat{\theta}^2$, $\hat{\eta}_0^2$, and $\hat{\eta}_{\hat{\lambda}}^2$. These are presented in Tables 1 through 5; the bold and italic faces, respectively, mean the biggest and second biggest values in each situation.

Please insert Tables 1 to 5 around here

From the results presented in these tables, we can see that, since there are nonlinear relationships between the datum as (A), (B), or (C), the NCCA and PNCCA methods always perform better than CCA. This reason is that CCA can not find a nonlinear relationship, since it evaluates the correlation values between linear combinations. The NCCA and PNCCA methods, in contrast to this, can find nonlinear relationships by using conversion functions. NCCA is almost always the best method since PNCCA is defined by adding the penalty term $\lambda\boldsymbol{P}$ to $\boldsymbol{S}_{ww}$ as $\boldsymbol{S}_{ww} + \lambda\boldsymbol{P}$ where $\lambda\boldsymbol{u}'\boldsymbol{P}\boldsymbol{u} \geq 0$ for any $\boldsymbol{u}$. However, recall that the NCCA has the critical issue that was pointed out by Hardoon, *et al.* (2004) and showed in (2.2) in the present paper. We considered using the PNCCA in order to avoid this issue.

Note that we considered the risk function in (3.1) in order to optimize λ based on the predictive values. Thus we also compared these methods by using the average value of $\hat{\boldsymbol{a}}_{\mathrm{C}}'\boldsymbol{\Sigma}_{y^*x^*}\hat{\boldsymbol{b}}_{\mathrm{C}}$, $\hat{\boldsymbol{a}}_{\mathrm{N}}'\boldsymbol{\Sigma}_{y^*w^*}\hat{\boldsymbol{d}}_{\mathrm{N}}$, and $\hat{\boldsymbol{a}}_{\mathrm{P}}'\boldsymbol{\Sigma}_{y^*w^*}\hat{\boldsymbol{d}}_{\mathrm{P}}$, and then we denoted the average value of each value as R_{C}^*, R_{N}^*, and R_{P}^* in Tables 6, 8, 10, 12, and 14. Moreover, we compared these methods by finding the average value of

$$\frac{\hat{\boldsymbol{a}}_{\mathrm{C}}'\boldsymbol{\Sigma}_{y^*x^*}\hat{\boldsymbol{b}}_{\mathrm{C}}}{\sqrt{\hat{\boldsymbol{a}}_{\mathrm{C}}'\boldsymbol{\Sigma}_{y^*y^*}\hat{\boldsymbol{a}}_{\mathrm{C}}\hat{\boldsymbol{b}}_{\mathrm{C}}'\boldsymbol{\Sigma}_{x^*x^*}\hat{\boldsymbol{b}}_{\mathrm{C}}}},\ \frac{\hat{\boldsymbol{a}}_{\mathrm{N}}'\boldsymbol{\Sigma}_{y^*w^*}\hat{\boldsymbol{d}}_{\mathrm{N}}}{\sqrt{\hat{\boldsymbol{a}}_{\mathrm{N}}'\boldsymbol{\Sigma}_{y^*y^*}\hat{\boldsymbol{a}}_{\mathrm{N}}\hat{\boldsymbol{d}}_{\mathrm{N}}'\boldsymbol{\Sigma}_{w^*w^*}\hat{\boldsymbol{d}}_{\mathrm{N}}}}\text{, and }\frac{\hat{\boldsymbol{a}}_{\mathrm{P}}'\boldsymbol{\Sigma}_{y^*w^*}\hat{\boldsymbol{d}}_{\mathrm{P}}}{\sqrt{\hat{\boldsymbol{a}}_{\mathrm{P}}'\boldsymbol{\Sigma}_{y^*y^*}\hat{\boldsymbol{a}}_{\mathrm{P}}\hat{\boldsymbol{d}}_{\mathrm{P}}'\boldsymbol{\Sigma}_{w^*w^*}\hat{\boldsymbol{d}}_{\mathrm{P}}}},$$

and we denoted the average of each value as R_{C}, R_{N}, and R_{P}, respectively, in Tables 7, 9, 11, 13, and 15. In Tables 6 to 15, the bold and italic faces mean the biggest and second biggest values, respectively, in each situation.

Please insert Tables 6 to 15 around here

Comparing R_{C}^* with R_{C}, R_{N}^* with R_{N}, and R_{P}^* with R_{P} in all situations, we note the results were similar except for the case where $p_0 = 3$. Hence, we focus on R_{C}^*, R_{N}^*, and R_{P}^* in Tables 6, 8, 10, 12, and 14.

First, we consider the results when using pattern (A), which are presented in Tables 6, 8, and 10. When ρ_x becomes large, the result values of CCA become small, the results of PNCCA become large, and the result values of NCCA also become large except when $(n, p_0) = (30, 5)$ and $(n, p_0) = (30, 8)$. The result values of each method become large in almost all cases when δ becomes large except when $(n, p_0) = (100, 3)$. In this pattern, we can change q_0. Thus, next, we consider the result values when q_0 changes. When q_0 changes from 3 to 8, the result values of NCCA become large in almost cases. When q_0 becomes large, the result values of PNCCA become large in almost situations in $(n, p_0, \delta) = (30, 3, 1)$, $(n, p_0, \delta) = (50, 3, 1)$, and $(n, p_0, \delta) = (100, 3, 1)$, and that of PNCCA become small in almost situations in $(n, p_0, \delta) = (30, 3, 3)$, $(n, p_0) = (30, 5)$, and $(n, p_0) = (30, 8)$. Next, we consider the results when p_0 becomes large. In $n = 50$ and $n = 100$, the result values of NCCA and PNCCA become large when p_0 becomes large. The result values of PNCCA also become large when $n = 30$ and p_0 becomes large. In this connection, we focus on the results when n becomes large. When n changes from 30 to 50, the result values of CCA almost all become small in $(p_0, q_0) = (3, 5)$ and $(p_0, q_0) = (3, 8)$, the result values of NCCA become large, and the result values of PNCCA become large when $p_0 = 3$ and $p_0 = 5$. When $(p_0, q_0) = (8, 5)$ and $(p_0, q_0) = (8, 8)$ and n changes from 30 to 50, the result values of PNCCA almost all become small. The result values of NCCA also become small when n changes from 30 to 50 in almost all situations when $(p_0, \rho_x) = (8, 0.8)$. When n changes from 50 to 100 and $p_0 = 8$, the result values of PNCCA almost always become small. The result values of PNCCA also become small when n changes from 30 to 100 except when $(p_0, \rho_x) = (8, 0.95)$. The result values of NCCA become large when n changes from 30 to 100.

Next, we consider the results when using pattern (B), which are presented in Table 12. When ρ_x becomes large, the result values of CCA become small when $(p_0, \delta) = (5, 1)$, $p_0 = 8$ except when $(n, \delta, p_0) = (30, 3, 3)$. When ρ_x becomes large, the result values of NCCA become large when $p_0 = 3$, $(n, p_0) = (100, 8)$, and $(n, \delta, p_0) = (50, 3, 8)$ but not when $(n, \delta, p_0) = (30, 1, 5)$. The result values of PNCCA become large when ρ_x becomes large. The result values of NCCA and PNCCA become large when δ becomes large. When δ becomes large, the result values of CCA also become large when $(n, p_0) = (30, 8)$, and that of CCA become small when $(n, p_0) = (50, 8)$ and $(n, p_0) = (100, 8)$. When $p_0 = 5$ and δ becomes large, the result values of CCA also almost all become small, except when $\rho_x = 0.8$. Next, we compare the result values when p_0 and n both become large. When p_0 becomes large, the result values of PNCCA become large. The result values of CCA become small when p_0 changes from 3 to 5 except when $(\delta, \rho_x) = (3, 0.8)$. When p_0 changes from 3 to 8, the result values of CCA almost all become small. The result values of NCCA become large when p_0 changes from 3 to 5 in $(n, \delta) = (50, 3)$ and $(n, \delta) = (100, 3)$, and when p_0 changes from 3 to 8 in $n = 50$ and $n = 100$. When p_0 changes from 5 to 8 and $\delta = 3$, the result values of NCCA almost all become large. In contrast to this, the result values of CCA become small when p_0 changes from 3 to 5 in $n = 30$. Moreover, when n changes from 50 to 100 and $p_0 = 3$, and it changes from 30 to 100 and $p_0 = 8$, the result values of CCA become small. The result values of NCCA become large except when $(p_0, \rho_x) = (3, 0.95)$ and $(p_0, \delta) = (8, 3)$, when n changes from 50 to 100. The result values of PNCCA almost always become large when n becomes large and $p_0 = 3$ and $p_0 = 5$. When $(p_0, \delta) = (8, 1)$ and n changes from 30 to 50 and 30 to 100, the result values of PNCCA also become large. When n changes from 50 to 100 and $p_0 = 8$, the result values of PNCCA become small.

Finally, we consider the results with pattern (C), which are in Table 14. When ρ_x becomes large, the result values of CCA and PNCCA become small and large, respectively. When ρ_x becomes large, the result values of NCCA almost always become large in $p_0 = 3$, $p_0 = 5$, $(n, p_0, \delta) = (50, 8, 3)$, and $(n, p_0) = (100, 8)$. When δ becomes large, the result values of NCCA and PNCCA become large. When δ becomes large, the result values of CCA become small when $(n, p_0) = (100, 8)$ and $p_0 = 3$, but not when $(p_0, \rho_x) = (3, 0.5)$. Next, we compare the results when p_0 becomes large and n becomes large. When ρ_x becomes large, the result values of PNCCA almost always become large. The result values of NCCA become small when p_0 changes from 3 to 5 and $n = 30$. Moreover, when n becomes large, the result values of NCCA and PNCCA almost always become large.

The PNCCA is always the best method except when $p_0 = 3$ and $(n, p_0) = (50, 5)$ in (A), when we compared R^*_{C}, R^*_{N}, and R^*_{P}. In addition to this, when we compared R_{C}, R_{N}, and R_{P}, the PNCCA is always the best method except when $p_0 = 3$. Further, of the three methods, PNCCA is never the worst. Based on these results, we recommend using the PNCCA method with the CV method, as proposed in this paper.

5. Conclusions

In the present paper, we considered finding a nonlinear relationship between random vectors by using PNCCA. This method is based on CCA (Hotelling, 1936) and a penalty method that is similar to the nonlinear regression model. The CCA method finds a linear relationship between random vectors, based on the correlation between linear combinations of them. The use of conversion functions allows a nonlinear relationship to be found by using the CCA method on the converted variables. Hardoon, *et al.* (2004) pointed out that this method has a critical problem, and, to avoid this, Akaho (2000) proposed the PNCCA when the conversion functions are the kernel functions. Although the result of PNCCA heavily depends on the penalty and several parameters, there have been no optimization methods proposed for them until the present paper. The reason for this is that the evaluation method for the covariance matrix is not defined.

In order to optimize the penalty and other parameters in the PNCCA method, we proposed using the CV method, which is based on the risk function in (3.1) in Section 3. Using the two samples $\{\boldsymbol{y}_i, \boldsymbol{w}_i\}$ and $\{\boldsymbol{y}_j, \boldsymbol{w}_j\}$, where $i \neq j$, we define $\hat{\boldsymbol{S}}_{[i,j]}$ for all i and j, $(i \neq j)$. On the other hands, for the fixed parameters, the results of PNCCA are derived based on the remains datum. Using $\hat{\boldsymbol{S}}_{[i,j]}$ and the results of PNCCA for each i and j, $(i \neq j)$, we then optimize the penalty and several parameters based on the CV method. Our numerical studies showed that the PNCCA method is almost always the best of the three we tested. Thus, we recommend using the PNCCA method, optimized by using the CV method.

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Appendix: Using proposed CV method to select variables in $\boldsymbol{y}$ and $\boldsymbol{x}$

Using the optimized penalty parameter $\hat{\lambda}$, the maximized value of $\boldsymbol{a}'\boldsymbol{\Sigma}_{yw}\boldsymbol{d}$ in (2.3) is estimated by using $\hat{\eta}_{\hat{\lambda}}$, which coincides with the square root of the largest eigenvalue of $(\boldsymbol{S}_{ww} + \hat{\lambda}\boldsymbol{P})^{-1}\boldsymbol{S}_{yw}'\boldsymbol{S}_{yy}^{-1}\boldsymbol{S}_{yw}$. In this section, we illustrate the variable selection method.

Let $\boldsymbol{y}^{[1]}$ and $\boldsymbol{x}^{[1]}$ be subsets of $\boldsymbol{y}$ and $\boldsymbol{x}$, respectively, and $\boldsymbol{w}^{[1]} = \boldsymbol{\psi}(\boldsymbol{x}^{[1]})$, where $\boldsymbol{\psi}(\cdot)$ is any conversion function that does not correspond with $\boldsymbol{\varphi}(\cdot)$. Let $\boldsymbol{S}_{w^{[1]}w^{[1]}}$, $\boldsymbol{S}_{y^{[1]}w^{[1]}}$, and $\boldsymbol{S}_{y^{[1]}y^{[1]}}$ be the sample covariance matrices of $\boldsymbol{w}^{[1]}$ and $\boldsymbol{w}^{[1]}$, $\boldsymbol{y}^{[1]}$ and $\boldsymbol{w}^{[1]}$, and $\boldsymbol{y}^{[1]}$ and $\boldsymbol{y}^{[1]}$, respectively, and let $\boldsymbol{P}^{[1]}$ be some known nonnegative penalty matrix. Based on the proposed CV method, the optimized penalty parameter $\hat{\lambda}^{[1]}$ is derived. We can then obtain $(\hat{\eta}^{[1]}_{\hat{\lambda}^{[1]}})^2$, which is the estimator of the maximized correlation between the linear combinations of $\boldsymbol{y}^{[1]}$ and $\boldsymbol{w}^{[1]}$.

Next, $(\hat{\eta}^{[2]}_{\hat{\lambda}^{[2]}})^2$ is derived using the same procedure as in the PNCCA method and the above procedure based on $\boldsymbol{y}^{[2]}$ and $\boldsymbol{x}^{[2]}$, where $\boldsymbol{y}^{[2]}$ and $\boldsymbol{x}^{[2]}$ are also subsets of $\boldsymbol{y}$ and $\boldsymbol{x}$ but are not the same as $\boldsymbol{y}^{[1]}$ and $\boldsymbol{x}^{[1]}$. If it holds that $\hat{\eta}^{[1]}_{\hat{\lambda}^{[1]}} > \hat{\eta}^{[2]}_{\hat{\lambda}^{[2]}}$ ($\hat{\eta}^{[1]}_{\hat{\lambda}^{[1]}} > 0$ and $\hat{\eta}^{[2]}_{\hat{\lambda}^{[2]}} > 0$), we select $\boldsymbol{y}^{[1]}$ and $\boldsymbol{x}^{[1]}$; $\boldsymbol{y}^{[2]}$ and $\boldsymbol{x}^{[2]}$ are selected if it does not hold.

Since we evaluate the covariance matrix by using the CV method, we conjecture that we may select another statistical estimation method based on the covariance matrix.

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Table 1: Average values of $\hat{\theta}^2$ (CCA), $\hat{\eta}_0^2$ (NCCA), and $\hat{\eta}_{\hat{\lambda}}^2$ (PNCCA) for $n = 30$ and (A)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
q_0	δ	ρ_x	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$
3	1	0.5	0.5409	**0.9468**	*0.9413*	0.5594	**0.9317**	*0.9106*	0.6683	**0.9663**	*0.9445*
		0.8	0.5122	**0.9610**	*0.9577*	0.5470	**0.9472**	*0.9315*	0.6700	**0.9739**	*0.9593*
		0.95	0.5343	**0.9680**	*0.9564*	0.5555	**0.9581**	*0.9390*	0.6852	**0.9763**	*0.9593*
	3	0.5	0.5498	**0.9813**	*0.9781*	0.5570	**0.9519**	*0.9348*	0.6646	**0.9738**	*0.9542*
		0.8	0.5103	**0.9906**	*0.9894*	0.5469	**0.9588**	*0.9448*	0.6701	**0.9771**	*0.9635*
		0.95	0.5350	**0.9895**	*0.9797*	0.5546	**0.9651**	*0.9470*	0.6844	**0.9780**	*0.9617*
5	1	0.5	0.6093	**0.9601**	*0.9544*	0.6508	**0.9456**	*0.9257*	0.7487	**0.9719**	*0.9505*
		0.8	0.5914	**0.9727**	*0.9704*	0.6475	**0.9561**	*0.9412*	0.7549	**0.9777**	*0.9640*
		0.95	0.6039	**0.9772**	*0.9675*	0.6512	**0.9645**	*0.9470*	0.7540	**0.9793**	*0.9626*
	3	0.5	0.6195	**0.9841**	*0.9810*	0.6499	**0.9571**	*0.9407*	0.7501	**0.9769**	*0.9580*
		0.8	0.5922	**0.9923**	*0.9912*	0.6423	**0.9633**	*0.9499*	0.7517	**0.9796**	*0.9666*
		0.95	0.6059	**0.9911**	*0.9830*	0.6490	**0.9688**	*0.9515*	0.7592	**0.9804**	*0.9642*
8	1	0.5	0.6987	**0.9724**	*0.9472*	0.7524	**0.9579**	*0.9358*	0.8366	**0.9785**	*0.9567*
		0.8	0.6825	**0.9823**	*0.9802*	0.7445	**0.9652**	*0.9513*	0.8350	**0.9821**	*0.9693*
		0.95	0.6887	**0.9841**	*0.9765*	0.7463	**0.9710**	*0.9541*	0.8396	**0.9831**	*0.9665*
	3	0.5	0.6957	**0.9870**	*0.9735*	0.7493	**0.9646**	*0.9463*	0.8389	**0.9815**	*0.9616*
		0.8	0.6856	**0.9940**	*0.9931*	0.7482	**0.9688**	*0.9562*	0.8344	**0.9836**	*0.9712*
		0.95	0.6950	**0.9928**	*0.9857*	0.7493	**0.9742**	*0.9584*	0.8381	**0.9838**	*0.9674*

Table 2: Average values of $\hat{\theta}^2$ (CCA), $\hat{\eta}_0^2$ (NCCA), and $\hat{\eta}_{\hat{\lambda}}^2$ (PNCCA) for $n = 50$ and (A)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
q_0	δ	ρ_x	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$
3	1	0.5	0.3522	**0.9509**	*0.9483*	0.4310	**0.9623**	*0.9587*	0.5016	**0.9842**	*0.9797*
		0.8	0.3316	**0.9671**	*0.9656*	0.4077	**0.9693**	*0.9668*	0.5157	**0.9928**	*0.9918*
		0.95	0.3305	**0.9740**	*0.9623*	0.4020	**0.9730**	*0.9711*	0.5353	**0.9953**	*0.9931*
	3	0.5	0.3563	**0.9914**	*0.9895*	0.4328	**0.9790**	*0.9763*	0.4962	**0.9922**	*0.9886*
		0.8	0.3343	**0.9939**	*0.9932*	0.4034	**0.9775**	*0.9755*	0.5127	**0.9960**	*0.9954*
		0.95	0.3267	**0.9943**	*0.9812*	0.4043	**0.9784**	*0.9769*	0.5332	**0.9969**	*0.9950*
5	1	0.5	0.4324	**0.9649**	*0.9625*	0.5080	**0.9693**	*0.9660*	0.5832	**0.9874**	*0.9834*
		0.8	0.4157	**0.9765**	*0.9750*	0.4933	**0.9730**	*0.9707*	0.5937	**0.9941**	*0.9933*
		0.95	0.4186	**0.9814**	*0.9711*	0.4885	**0.9757**	*0.9739*	0.6000	**0.9960**	*0.9940*
	3	0.5	0.4298	**0.9933**	*0.9915*	0.5077	**0.9806**	*0.9780*	0.5788	**0.9929**	*0.9895*
		0.8	0.4168	**0.9952**	*0.9945*	0.4919	**0.9789**	*0.9770*	0.5892	**0.9963**	*0.9957*
		0.95	0.4144	**0.9953**	*0.9844*	0.4915	**0.9797**	*0.9781*	0.5996	**0.9972**	*0.9953*
8	1	0.5	0.5194	**0.9761**	*0.9740*	0.5916	**0.9750**	*0.9720*	0.6610	**0.9902**	*0.9865*
		0.8	0.5107	**0.9838**	*0.9826*	0.5816	**0.9768**	*0.9746*	0.6681	**0.9953**	*0.9945*
		0.95	0.5029	**0.9871**	*0.9783*	0.5802	**0.9789**	*0.9770*	0.6759	**0.9967**	*0.9948*
	3	0.5	0.5141	**0.9948**	*0.9932*	0.5941	**0.9825**	*0.9798*	0.6646	**0.9936**	*0.9904*
		0.8	0.5105	**0.9961**	*0.9953*	0.5847	**0.9808**	*0.9788*	0.6712	**0.9967**	*0.9961*
		0.95	0.5064	**0.9961**	*0.9875*	0.5779	**0.9811**	*0.9795*	0.6762	**0.9974**	*0.9957*

Table 3: Average values of $\hat{\theta}^2$ (CCA), $\hat{\eta}_0^2$ (NCCA), and $\hat{\eta}_{\hat{\lambda}}^2$ (PNCCA) for $n = 100$ and (A)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
q_0	δ	ρ_x	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$
3	1	0.5	0.2504	**0.9461**	*0.9456*	0.3070	**0.9785**	*0.9777*	0.3618	**0.9829**	*0.9802*
		0.8	0.2255	**0.9560**	*0.9558*	0.2805	**0.9892**	*0.9889*	0.3542	**0.9916**	*0.9905*
		0.95	0.2198	**0.9606**	*0.9522*	0.2742	**0.9909**	*0.9899*	0.3554	**0.9941**	*0.9932*
	3	0.5	0.2515	**0.9798**	*0.9797*	0.3046	**0.9925**	*0.9921*	0.3586	**0.9919**	*0.9897*
		0.8	*0.2252*	**0.9780**	**0.9780**	0.2788	**0.9960**	*0.9959*	0.3552	**0.9961**	*0.9952*
		0.95	0.2222	**0.9771**	*0.9702*	0.2768	**0.9950**	*0.9942*	0.3558	**0.9964**	*0.9957*
5	1	0.5	0.3073	**0.9574**	*0.9570*	0.3610	**0.9834**	*0.9827*	0.4192	**0.9860**	*0.9836*
		0.8	0.2905	**0.9638**	*0.9636*	0.3462	**0.9915**	*0.9913*	0.4146	**0.9932**	*0.9922*
		0.95	0.2852	**0.9664**	*0.9599*	0.3404	**0.9924**	*0.9914*	0.4147	**0.9949**	*0.9941*
	3	0.5	0.3054	**0.9814**	*0.9813*	0.3605	**0.9932**	*0.9928*	0.4231	**0.9924**	*0.9903*
		0.8	*0.2895*	**0.9793**	**0.9793**	0.3455	**0.9963**	*0.9962*	0.4142	**0.9964**	*0.9955*
		0.95	0.2820	**0.9782**	*0.9728*	0.3421	**0.9953**	*0.9944*	0.4166	**0.9966**	*0.9959*
8	1	0.5	0.3703	**0.9667**	*0.9665*	0.4221	**0.9870**	*0.9865*	0.4794	**0.9886**	*0.9863*
		0.8	0.3551	**0.9701**	*0.9699*	0.4132	**0.9934**	*0.9932*	0.4812	**0.9945**	*0.9935*
		0.95	0.3566	**0.9714**	*0.9652*	0.4079	**0.9936**	*0.9924*	0.4767	**0.9956**	*0.9948*
	3	0.5	0.3672	**0.9830**	*0.9829*	0.4226	**0.9937**	*0.9933*	0.4795	**0.9929**	*0.9908*
		0.8	0.3541	**0.9807**	*0.9806*	0.4141	**0.9966**	*0.9965*	0.4777	**0.9966**	*0.9958*
		0.95	0.3559	**0.9794**	*0.9747*	0.4096	**0.9955**	*0.9945*	0.4772	**0.9968**	*0.9960*

Table 4: Average values of $\hat{\theta}^2$ (CCA), $\hat{\eta}_0^2$ (NCCA), and $\hat{\eta}_{\hat{\lambda}}^2$ (PNCCA) for (B)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
n	δ	ρ_x	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$
30	1	0.5	0.5082	**0.6672**	*0.5687*	0.5448	**0.6735**	*0.5453*	*0.6357*	**0.7816**	0.6339
		0.8	0.4487	**0.6824**	*0.6271*	0.5396	**0.7411**	*0.6635*	0.6582	**0.8323**	*0.7507*
		0.95	0.4515	**0.7614**	*0.7083*	0.5426	**0.7831**	*0.7128*	0.6722	**0.8632**	*0.7696*
	3	0.5	0.5343	**0.8477**	*0.8157*	0.5197	**0.8567**	*0.8175*	0.6396	**0.9249**	*0.8870*
		0.8	0.5001	**0.8903**	*0.8673*	0.5058	**0.9014**	*0.8768*	0.6373	**0.9441**	*0.9225*
		0.95	0.4680	**0.9390**	*0.9215*	0.5702	**0.9295**	*0.9046*	0.6628	**0.9577**	*0.9316*
50	1	0.5	0.3530	**0.4971**	*0.4252*	0.4319	**0.6029**	*0.5233*	0.4907	**0.7027**	*0.6264*
		0.8	0.3198	**0.5423**	*0.4894*	0.4460	**0.6929**	*0.6648*	0.4919	**0.8053**	*0.7709*
		0.95	0.3239	**0.6151**	*0.5486*	0.4431	**0.8298**	*0.8090*	0.5158	**0.8709**	*0.8451*
	3	0.5	0.3413	**0.7422**	*0.7267*	0.4247	**0.8522**	*0.8420*	0.4113	**0.9197**	*0.9063*
		0.8	0.2772	**0.7872**	*0.7748*	0.4534	**0.9157**	*0.9095*	0.4396	**0.9633**	*0.9587*
		0.95	0.2967	**0.8562**	*0.8180*	0.5195	**0.9524**	*0.9489*	0.4874	**0.9764**	*0.9714*
100	1	0.5	0.2445	**0.4398**	*0.4046*	0.3208	**0.5352**	*0.5100*	0.3715	**0.5574**	*0.5015*
		0.8	0.2324	**0.4930**	*0.4704*	0.2793	**0.6912**	*0.6736*	0.3534	**0.6758**	*0.6482*
		0.95	0.2619	**0.5518**	*0.5151*	0.2684	**0.7712**	*0.7449*	0.3325	**0.7791**	*0.7562*
	3	0.5	0.2496	**0.7066**	*0.7007*	0.3288	**0.8455**	*0.8407*	0.3859	**0.8547**	*0.8412*
		0.8	0.2007	**0.7570**	*0.7525*	0.2615	**0.9306**	*0.9282*	0.3383	**0.9292**	*0.9245*
		0.95	0.2724	**0.8112**	*0.7806*	0.2145	**0.9502**	*0.9456*	0.2861	**0.9559**	*0.9524*

Table 5: Average values of $\hat{\theta}^2$ (CCA), $\hat{\eta}_0^2$ (NCCA), and $\hat{\eta}_{\hat{\lambda}}^2$ (PNCCA) for (C)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
n	δ	ρ_x	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$	$\hat{\theta}^2$	$\hat{\eta}_0^2$	$\hat{\eta}_{\hat{\lambda}}^2$
30	1	0.5	0.5468	**0.7017**	*0.6122*	*0.5952*	**0.7149**	0.5889	*0.6927*	**0.8101**	0.6608
		0.8	0.5155	**0.7062**	*0.6528*	0.5947	**0.7589**	*0.6740*	0.7033	**0.8526**	*0.7683*
		0.95	0.5089	**0.7855**	*0.7340*	0.5938	**0.8030**	*0.7330*	0.7143	**0.8749**	*0.7838*
	3	0.5	0.5728	**0.8835**	*0.8619*	0.5796	**0.8773**	*0.8388*	0.6916	**0.9309**	*0.8955*
		0.8	0.5665	**0.9103**	*0.8945*	0.5738	**0.9096**	*0.8843*	0.6981	**0.9484**	*0.9273*
		0.95	0.5100	**0.9478**	*0.9343*	0.6103	**0.9346**	*0.9108*	0.7229	**0.9595**	*0.9289*
50	1	0.5	0.3895	**0.5577**	*0.4823*	0.4708	**0.6383**	*0.5612*	0.5299	**0.7240**	*0.6460*
		0.8	0.3683	**0.5825**	*0.5378*	0.4847	**0.7216**	*0.6928*	0.5321	**0.8194**	*0.7876*
		0.95	0.3739	**0.6515**	*0.5836*	0.4857	**0.8389**	*0.8193*	0.5529	**0.8768**	*0.8515*
	3	0.5	0.3802	**0.8202**	*0.8099*	0.4632	**0.8782**	*0.8693*	0.4829	**0.9298**	*0.9171*
		0.8	0.3276	**0.8497**	*0.8424*	0.4985	**0.9285**	*0.9239*	0.4990	**0.9667**	*0.9625*
		0.95	0.3545	**0.8913**	*0.8617*	0.5755	**0.9585**	*0.9556*	0.5409	**0.9788**	*0.9744*
100	1	0.5	0.2777	**0.4956**	*0.4670*	0.3499	**0.5742**	*0.5486*	0.4041	**0.5759**	*0.5224*
		0.8	0.2650	**0.5282**	*0.5121*	0.3177	**0.7184**	*0.7019*	0.3866	**0.6968**	*0.6708*
		0.95	0.2922	**0.5813**	*0.5462*	0.3043	**0.7824**	*0.7590*	0.3653	**0.7899**	*0.7683*
	3	0.5	0.2787	**0.7917**	*0.7881*	0.3553	**0.8723**	*0.8672*	0.4038	**0.8707**	*0.8585*
		0.8	0.2382	**0.8153**	*0.8119*	0.3077	**0.9392**	*0.9371*	0.3678	**0.9371**	*0.9326*
		0.95	0.2960	**0.8547**	*0.8368*	0.2714	**0.9557**	*0.9497*	0.3329	**0.9609**	*0.9576*

Table 6: Average values of R^*_{C} (CCA), R^*_{N} (NCCA), and R^*_{P} (PNCCA) for $n = 30$ and (A)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
q_0	δ	ρ_x	R^*_{C}	R^*_{N}	R^*_{P}	R^*_{C}	R^*_{N}	R^*_{P}	R^*_{C}	R^*_{N}	R^*_{P}
3	1	0.5	0.0055	*1.0848*	**1.0888**	0.0028	*1.0723*	**1.2443**	0.0070	*1.4206*	**1.6244**
		0.8	0.0029	*1.6265*	**1.6713**	0.0017	*1.6642*	**2.4493**	0.0020	*1.5904*	**3.5193**
		0.95	0.0021	**2.3404**	*2.2292*	0.0007	*0.8485*	**3.6166**	0.0010	*0.5696*	**5.8488**
	3	0.5	0.0156	*1.1349*	**1.1383**	0.0033	*1.0898*	**1.2380**	0.0063	*1.4944*	**1.6558**
		0.8	0.0019	*1.6746*	**1.6993**	0.0027	*1.6857*	**2.4588**	0.0032	*1.6175*	**3.5545**
		0.95	0.0012	**2.4598**	*2.2841*	0.0014	*0.8380*	**3.6303**	0.0006	*0.5747*	**5.9185**
5	1	0.5	0.0101	**1.0912**	*1.0904*	0.0040	*1.0728*	**1.2353**	0.0048	*1.4135*	**1.6298**
		0.8	0.0027	*1.6389*	**1.6753**	0.0026	*1.6435*	**2.4555**	0.0024	*1.5689*	**3.4765**
		0.95	0.0013	**2.3832**	*2.2656*	0.0009	*0.8670*	**3.5930**	0.0006	*0.5830*	**5.7756**
	3	0.5	0.0075	*1.1186*	**1.1206**	0.0041	*1.0938*	**1.2437**	0.0049	*1.4588*	**1.6316**
		0.8	0.0052	*1.6788*	**1.7040**	0.0024	*1.6680*	**2.4355**	0.0022	*1.6021*	**3.5511**
		0.95	0.0010	**2.4589**	*2.3020*	0.0007	*0.8388*	**3.5998**	0.0009	*0.5777*	**5.8660**
8	1	0.5	0.0068	**1.0992**	*1.0367*	0.0051	*1.0519*	**1.2061**	0.0052	*1.3710*	**1.5844**
		0.8	0.0055	*1.6640*	**1.6981**	0.0032	*1.6006*	**2.4153**	0.0032	*1.5513*	**3.4621**
		0.95	0.0017	**2.3961**	*2.2814*	0.0008	*0.8437*	**3.5642**	0.0007	*0.5925*	**5.6960**
	3	0.5	0.0065	**1.1145**	*1.0829*	0.0045	*1.0623*	**1.2186**	0.0052	*1.3998*	**1.5976**
		0.8	0.0037	*1.6753*	**1.6990**	0.0024	*1.6347*	**2.4313**	0.0024	*1.5424*	**3.4618**
		0.95	0.0020	**2.4499**	*2.2716*	0.0008	*0.8474*	**3.5708**	0.0005	*0.5795*	**5.7208**

Table 7: Average values of R_{C} (CCA), R_{N} (NCCA), and R_{P} (PNCCA) for $n = 30$ and (A)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
q_0	δ	ρ_x	R_{C}	R_{N}	R_{P}	R_{C}	R_{N}	R_{P}	R_{C}	R_{N}	R_{P}
3	1	0.5	0.0070	**0.9477**	*0.9474*	0.0054	*0.8811*	**0.9430**	0.0102	*0.9111*	**0.9704**
		0.8	0.0068	*0.9594*	**0.9657**	0.0061	*0.9102*	**0.9689**	0.0073	*0.8998*	**0.9760**
		0.95	0.0121	**0.9649**	*0.8896*	0.0082	*0.8796*	**0.9714**	0.0129	*0.9025*	**0.9778**
	3	0.5	0.0196	*0.9846*	**0.9849**	0.0071	*0.9080*	**0.9631**	0.0101	*0.9356*	**0.9820**
		0.8	0.0042	*0.9868*	**0.9879**	0.0094	*0.9197*	**0.9781**	0.0098	*0.9072*	**0.9814**
		0.95	0.0096	**0.9866**	*0.9086*	0.0151	*0.8829*	**0.9735**	0.0083	*0.9080*	**0.9822**
5	1	0.5	0.0121	**0.9545**	*0.9513*	0.0077	*0.8838*	**0.9419**	0.0075	*0.9058*	**0.9655**
		0.8	0.0058	*0.9652*	**0.9702**	0.0090	*0.9014*	**0.9655**	0.0077	*0.8894*	**0.9723**
		0.95	0.0059	**0.9705**	*0.8990*	0.0098	*0.8766*	**0.9645**	0.0076	*0.8936*	**0.9708**
	3	0.5	0.0093	**0.9838**	*0.9829*	0.0075	*0.9044*	**0.9582**	0.0072	*0.9259*	**0.9772**
		0.8	0.0111	*0.9872*	**0.9881**	0.0076	*0.9137*	**0.9734**	0.0072	*0.8985*	**0.9779**
		0.95	0.0063	**0.9868**	*0.9164*	0.0073	*0.8819*	**0.9705**	0.0120	*0.9002*	**0.9745**
8	1	0.5	0.0083	**0.9568**	*0.9037*	0.0087	*0.8696*	**0.9229**	0.0075	*0.8900*	**0.9455**
		0.8	0.0104	*0.9686*	**0.9721**	0.0093	*0.8930*	**0.9585**	0.0099	*0.8784*	**0.9641**
		0.95	0.0081	**0.9710**	*0.9022*	0.0092	*0.8728*	**0.9554**	0.0091	*0.8802*	**0.9570**
	3	0.5	0.0077	**0.9804**	*0.9514*	0.0080	*0.8866*	**0.9412**	0.0075	*0.9037*	**0.9568**
		0.8	0.0086	*0.9879*	**0.9885**	0.0077	*0.9005*	**0.9665**	0.0079	*0.8818*	**0.9673**
		0.95	0.0086	**0.9831**	*0.9045*	0.0085	*0.8730*	**0.9595**	0.0068	*0.8815*	**0.9596**

Table 8: Average values of R^*_{C} (CCA), R^*_{N} (NCCA), and R^*_{P} (PNCCA) for $n = 50$ and (A)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
q_0	δ	ρ_x	R^*_{C}	R^*_{N}	R^*_{P}	R^*_{C}	R^*_{N}	R^*_{P}	R^*_{C}	R^*_{N}	R^*_{P}
3	1	0.5	0.0046	**1.1368**	*1.1325*	0.0055	**1.3697**	*1.3605*	0.0042	*1.5363*	**1.5706**
		0.8	0.0028	**1.8384**	*1.8374*	0.0020	*2.6066*	**2.6390**	0.0025	*3.4349*	**3.5174**
		0.95	0.0025	*2.4768*	**2.4821**	0.0007	*3.5793*	**4.0919**	0.0006	*5.1110*	**6.2173**
	3	0.5	0.0070	**1.1820**	*1.1759*	0.0037	**1.3715**	*1.3574*	0.0061	*1.5809*	**1.6030**
		0.8	0.0042	**1.8803**	*1.8716*	0.0038	**2.6648**	*2.6590*	0.0019	*3.5191*	**3.5492**
		0.95	0.0011	**2.5765**	*2.5221*	0.0007	*3.7883*	**4.1181**	0.0004	*5.4452*	**6.2972**
5	1	0.5	0.0069	**1.1455**	*1.1404*	0.0055	**1.3621**	*1.3533*	0.0045	*1.5577*	**1.5855**
		0.8	0.0032	**1.8480**	*1.8430*	0.0025	*2.6401*	**2.6667**	0.0021	*3.4211*	**3.4885**
		0.95	0.0013	*2.5102*	**2.5123**	0.0009	*3.5909*	**4.1073**	0.0005	*5.1608*	**6.2080**
	3	0.5	0.0074	**1.1671**	*1.1607*	0.0047	**1.3896**	*1.3751*	0.0053	*1.5632*	**1.5857**
		0.8	0.0041	**1.8859**	*1.8769*	0.0032	**2.6548**	*2.6521*	0.0019	*3.5276*	**3.5593**
		0.95	0.0013	**2.5805**	*2.5345*	0.0007	*3.7451*	**4.1177**	0.0006	*5.4121*	**6.2852**
8	1	0.5	0.0060	**1.1588**	*1.1532*	0.0076	**1.3608**	*1.3512*	0.0051	*1.5440*	**1.5711**
		0.8	0.0043	**1.8764**	*1.8695*	0.0028	*2.6122*	**2.6418**	0.0024	*3.4513*	**3.5162**
		0.95	0.0016	**2.5381**	*2.5289*	0.0008	*3.5632*	**4.0960**	0.0006	*5.1504*	**6.2035**
	3	0.5	0.0063	**1.1669**	*1.1596*	0.0057	**1.3798**	*1.3646*	0.0052	*1.5522*	**1.5769**
		0.8	0.0039	**1.8831**	*1.8726*	0.0028	*2.6527*	**2.6566**	0.0023	*3.4788*	**3.5145**
		0.95	0.0018	**2.5776**	*2.5306*	0.0008	*3.7039*	**4.1246**	0.0005	*5.3411*	**6.2406**

Table 9: Average values of R_{C} (CCA), R_{N} (NCCA), and R_{P} (PNCCA) for $n = 50$ and (A)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
q_0	δ	ρ_x	R_{C}	R_{N}	R_{P}	R_{C}	R_{N}	R_{P}	R_{C}	R_{N}	R_{P}
3	1	0.5	0.0062	*0.9531*	**0.9549**	0.0070	*0.9650*	**0.9695**	0.0068	*0.9708*	**0.9787**
		0.8	*0.0050*	**0.9670**	**0.9670**	0.0054	*0.9764*	**0.9798**	0.0105	*0.9838*	**0.9858**
		0.95	0.0125	**0.9700**	*0.9454*	0.0078	*0.9796*	**0.9816**	0.0113	*0.9860*	**0.9886**
	3	0.5	0.0091	*0.9866*	**0.9876**	0.0052	*0.9838*	**0.9874**	0.0098	*0.9831*	**0.9896**
		0.8	0.0073	*0.9880*	**0.9883**	0.0097	*0.9858*	**0.9882**	0.0080	*0.9891*	**0.9901**
		0.95	0.0076	**0.9871**	*0.9587*	0.0081	*0.9823*	**0.9838**	0.0074	*0.9901*	**0.9921**
5	1	0.5	0.0082	*0.9611*	**0.9627**	0.0078	*0.9687*	**0.9736**	0.0070	*0.9723*	**0.9805**
		0.8	0.0056	**0.9737**	*0.9731*	0.0072	*0.9777*	**0.9809**	0.0079	*0.9851*	**0.9868**
		0.95	0.0064	**0.9758**	*0.9529*	0.0091	*0.9797*	**0.9818**	0.0082	*0.9860*	**0.9885**
	3	0.5	0.0097	*0.9871*	**0.9882**	0.0063	*0.9836*	**0.9874**	0.0079	*0.9838*	**0.9899**
		0.8	0.0077	*0.9885*	**0.9888**	0.0082	*0.9854*	**0.9882**	0.0072	*0.9904*	**0.9914**
		0.95	0.0094	**0.9880**	*0.9634*	0.0070	*0.9842*	**0.9858**	0.0092	*0.9882*	**0.9903**
8	1	0.5	0.0082	*0.9665*	**0.9681**	0.0096	*0.9707*	**0.9756**	0.0075	*0.9747*	**0.9823**
		0.8	0.0087	**0.9778**	*0.9771*	0.0074	*0.9765*	**0.9803**	0.0086	*0.9857*	**0.9876**
		0.95	0.0097	**0.9781**	*0.9551*	0.0090	*0.9800*	**0.9823**	0.0081	*0.9845*	**0.9868**
	3	0.5	0.0080	*0.9863*	**0.9872**	0.0077	*0.9819*	**0.9860**	0.0078	*0.9807*	**0.9879**
		0.8	0.0090	**0.9902**	*0.9900*	0.0079	*0.9823*	**0.9854**	0.0080	*0.9886*	**0.9900**
		0.95	0.0087	**0.9852**	*0.9599*	0.0088	*0.9829*	**0.9850**	0.0063	*0.9884*	**0.9905**

Table 10: Average values of R^*_{C} (CCA), R^*_{N} (NCCA), and R^*_{P} (PNCCA) for $n = 100$ and (A)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
q_0	δ	ρ_x	R^*_{C}	R^*_{N}	R^*_{P}	R^*_{C}	R^*_{N}	R^*_{P}	R^*_{C}	R^*_{N}	R^*_{P}
3	1	0.5	0.0055	*1.2018*	**1.2057**	0.0032	*1.5306*	**1.5352**	0.0041	*1.5487*	**1.5695**
		0.8	0.0029	*1.9092*	**1.9138**	0.0021	*2.8659*	**2.8950**	0.0023	*3.2276*	**3.2883**
		0.95	0.0021	**2.5552**	*2.5396*	0.0007	*3.9258*	**4.2784**	0.0007	*5.5941*	**6.0614**
	3	0.5	0.0067	*1.2531*	**1.2561**	0.0043	*1.5357*	**1.5381**	0.0065	*1.5865*	**1.6023**
		0.8	0.0045	**1.9513**	*1.9497*	0.0029	*2.9057*	**2.9187**	0.0023	*3.2731*	**3.3141**
		0.95	0.0014	**2.6314**	*2.5962*	0.0006	*4.0318*	**4.3034**	0.0005	*5.7878*	**6.1337**
5	1	0.5	0.0062	*1.2067*	**1.2105**	0.0047	*1.5263*	**1.5302**	0.0051	*1.5673*	**1.5850**
		0.8	0.0033	*1.9120*	**1.9155**	0.0024	*2.9000*	**2.9256**	0.0021	*3.2101*	**3.2612**
		0.95	0.0015	**2.5791**	*2.5716*	0.0008	*3.9569*	**4.2926**	0.0005	*5.6221*	**6.0548**
	3	0.5	0.0068	*1.2363*	**1.2393**	0.0042	*1.5575*	**1.5604**	0.0055	*1.5703*	**1.5851**
		0.8	0.0034	**1.9561**	*1.9548*	0.0028	*2.9022*	**2.9150**	0.0019	*3.2823*	**3.3243**
		0.95	0.0017	**2.6327**	*2.6061*	0.0007	*4.0373*	**4.3078**	0.0007	*5.7813*	**6.1237**
8	1	0.5	0.0062	*1.2187*	**1.2221**	0.0061	*1.5266*	**1.5304**	0.0053	*1.5545*	**1.5720**
		0.8	0.0041	*1.9385*	**1.9404**	0.0027	*2.8822*	**2.9037**	0.0024	*3.2381*	**3.2901**
		0.95	0.0020	**2.5963**	*2.5784*	0.0008	*3.9735*	**4.2905**	0.0006	*5.6429*	**6.0518**
	3	0.5	0.0060	*1.2348*	**1.2378**	0.0056	*1.5482*	**1.5508**	0.0048	*1.5607*	**1.5771**
		0.8	0.0043	**1.9512**	*1.9502*	0.0029	*2.9138*	**2.9264**	0.0024	*3.2405*	**3.2813**
		0.95	0.0016	**2.6274**	*2.5991*	0.0009	*4.0515*	**4.3232**	0.0005	*5.7381*	**6.0822**

Table 11: Average values of R_{C} (CCA), R_{N} (NCCA), and R_{P} (PNCCA) for $n = 100$ and (A)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
q_0	δ	ρ_x	R_{C}	R_{N}	R_{P}	R_{C}	R_{N}	R_{P}	R_{C}	R_{N}	R_{P}
3	1	0.5	0.0076	*0.9557*	**0.9561**	0.0052	*0.9708*	**0.9716**	0.0062	*0.9764*	**0.9798**
		0.8	0.0052	*0.9684*	**0.9687**	0.0060	*0.9815*	**0.9821**	0.0091	*0.9862*	**0.9875**
		0.95	0.0110	**0.9706**	*0.9529*	0.0077	*0.9825*	**0.9836**	0.0111	*0.9883*	**0.9893**
	3	0.5	0.0091	*0.9874*	**0.9875**	0.0074	*0.9891*	**0.9894**	0.0100	*0.9876*	**0.9902**
		0.8	*0.0073*	**0.9882**	**0.9882**	0.0080	*0.9899*	**0.9901**	0.0085	*0.9907*	**0.9918**
		0.95	0.0078	**0.9870**	*0.9721*	0.0076	*0.9845*	**0.9855**	0.0074	*0.9918*	**0.9926**
5	1	0.5	0.0076	*0.9634*	**0.9638**	0.0077	*0.9768*	**0.9775**	0.0077	*0.9791*	**0.9822**
		0.8	0.0054	*0.9747*	**0.9749**	0.0069	*0.9835*	**0.9839**	0.0071	*0.9877*	**0.9888**
		0.95	0.0065	**0.9763**	*0.9621*	0.0088	*0.9835*	**0.9845**	0.0076	*0.9885*	**0.9894**
	3	0.5	0.0089	*0.9879*	**0.9880**	0.0065	*0.9901*	**0.9906**	0.0080	*0.9885*	**0.9910**
		0.8	*0.0069*	**0.9886**	**0.9886**	0.0076	*0.9911*	**0.9912**	0.0074	*0.9922*	**0.9932**
		0.95	0.0096	**0.9876**	*0.9760*	0.0067	*0.9876*	**0.9885**	0.0100	*0.9902*	**0.9910**
8	1	0.5	0.0082	*0.9693*	**0.9696**	0.0089	*0.9804*	**0.9810**	0.0077	*0.9815*	**0.9849**
		0.8	0.0084	*0.9788*	**0.9790**	0.0074	*0.9853*	**0.9857**	0.0088	*0.9886*	**0.9899**
		0.95	0.0097	**0.9781**	*0.9632*	0.0087	*0.9866*	**0.9873**	0.0079	*0.9873*	**0.9880**
	3	0.5	0.0077	*0.9867*	**0.9868**	0.0079	*0.9898*	**0.9904**	0.0070	*0.9865*	**0.9897**
		0.8	0.0096	*0.9897*	**0.9898**	0.0078	*0.9900*	**0.9902**	0.0084	*0.9908*	**0.9920**
		0.95	0.0084	**0.9844**	*0.9718*	0.0085	*0.9886*	**0.9891**	0.0068	*0.9909*	**0.9917**

Table 12: Average values of R^*_{C} (CCA), R^*_{N} (NCCA), and R^*_{P} (PNCCA) for (B)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
n	δ	ρ_x	R^*_{C}	R^*_{N}	R^*_{P}	R^*_{C}	R^*_{N}	R^*_{P}	R^*_{C}	R^*_{N}	R^*_{P}
30	1	0.5	0.0050	*0.1803*	**0.1924**	0.0060	*0.1426*	**0.2362**	0.0042	*0.1415*	**0.3359**
		0.8	0.0029	*0.3026*	**0.3593**	0.0017	*0.1841*	**0.5203**	0.0020	*0.2017*	**1.0257**
		0.95	0.0027	*0.5701*	**0.7467**	0.0014	*0.2478*	**1.3996**	0.0005	*0.1366*	**2.7816**
	3	0.5	0.0088	**0.5705**	*0.5645*	0.0025	*0.5279*	**0.6730**	0.0049	*0.8081*	**1.0558**
		0.8	0.0015	*0.9216*	**0.9264**	0.0036	*0.8489*	**1.5573**	0.0026	*0.9735*	**2.3869**
		0.95	0.0038	**1.5283**	*1.5091*	0.0025	*0.6209*	**2.5992**	0.0007	*0.3815*	**4.4913**
50	1	0.5	0.0051	*0.1904*	**0.1936**	0.0048	*0.1715*	**0.2568**	0.0051	*0.2230*	**0.3839**
		0.8	0.0032	*0.3164*	**0.4315**	0.0020	*0.3590*	**0.6171**	0.0021	*0.5467*	**1.1514**
		0.95	0.0023	*0.4452*	**0.8387**	0.0011	*0.6805*	**1.7296**	0.0005	*0.6244*	**3.3308**
	3	0.5	0.0053	*0.6274*	**0.6285**	0.0027	*0.6998*	**0.7309**	0.0033	*0.9603*	**1.0420**
		0.8	0.0025	*1.0326*	**1.1067**	0.0060	*1.5704*	**1.7519**	0.0012	*2.1235*	**2.4541**
		0.95	0.0011	*1.4164*	**1.6434**	0.0010	*2.2205*	**2.9867**	0.0005	*2.8972*	**5.0905**
100	1	0.5	0.0051	*0.2578*	**0.2644**	0.0075	*0.2691*	**0.3349**	0.0037	*0.2601*	**0.3816**
		0.8	0.0030	*0.4277*	**0.5193**	0.0014	*0.4836*	**0.7042**	0.0019	*0.6373*	**1.0948**
		0.95	0.0014	*0.5038*	**0.9007**	0.0007	*0.8114*	**1.7327**	0.0005	*0.9769*	**3.3225**
	3	0.5	0.0044	*0.6661*	**0.6800**	0.0030	*0.7958*	**0.8217**	0.0029	*0.9588*	**1.0304**
		0.8	0.0017	*1.1146*	**1.1567**	0.0022	*1.8378*	**1.9598**	0.0017	*2.0936*	**2.3122**
		0.95	0.0010	*1.3845*	**1.6391**	0.0006	*2.4001*	**3.1127**	0.0005	*3.3980*	**4.9799**

Table 13: Average values of R_{C} (CCA), R_{N} (NCCA), and R_{P} (PNCCA) and (B)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
n	δ	ρ_x	R_{C}	R_{N}	R_{P}	R_{C}	R_{N}	R_{P}	R_{C}	R_{N}	R_{P}
30	1	0.5	0.0066	*0.1775*	**0.1861**	0.0084	*0.1375*	**0.1801**	0.0062	*0.1291*	**0.2039**
		0.8	0.0050	*0.2333*	**0.2481**	0.0059	*0.1558*	**0.2255**	0.0071	*0.2063*	**0.3223**
		0.95	0.0108	**0.3582**	*0.3550*	0.0093	*0.3434*	**0.4088**	0.0070	*0.4026*	**0.5196**
	3	0.5	0.0116	**0.5377**	*0.5259*	0.0041	*0.4626*	**0.5170**	0.0064	*0.5600*	**0.6291**
		0.8	0.0024	**0.5889**	*0.5772*	0.0157	*0.5805*	**0.6510**	0.0099	*0.6010*	**0.6828**
		0.95	0.0194	**0.6695**	*0.6366*	0.0082	*0.6211*	**0.6991**	0.0091	*0.7010*	**0.7886**
50	1	0.5	0.0069	**0.1746**	*0.1741*	0.0060	*0.1553*	**0.1910**	0.0081	*0.1806*	**0.2383**
		0.8	0.0059	*0.2399*	**0.2619**	0.0045	*0.2197*	**0.2400**	0.0084	*0.3078*	**0.3381**
		0.95	0.0112	*0.3557*	**0.3795**	0.0077	*0.4322*	**0.4427**	0.0083	*0.5402*	**0.5673**
	3	0.5	0.0073	*0.5383*	**0.5387**	0.0028	*0.5229*	**0.5324**	0.0062	*0.6266*	**0.6436**
		0.8	0.0043	*0.5813*	**0.5908**	0.0093	*0.6414*	**0.6499**	0.0062	*0.6845*	**0.6928**
		0.95	0.0101	**0.6862**	*0.6686*	0.0060	*0.7198*	**0.7225**	0.0107	*0.8075*	**0.8146**
100	1	0.5	0.0068	**0.2287**	*0.2286*	0.0087	*0.2048*	**0.2243**	0.0059	*0.2041*	**0.2439**
		0.8	0.0060	*0.2858*	**0.2983**	0.0047	*0.2445*	**0.2518**	0.0071	*0.3192*	**0.3358**
		0.95	0.0089	*0.3753*	**0.3857**	0.0070	*0.4588*	**0.4602**	0.0069	*0.5595*	**0.5688**
	3	0.5	0.0063	*0.5517*	**0.5554**	0.0032	*0.5291*	**0.5343**	0.0041	*0.6338*	**0.6481**
		0.8	0.0042	*0.6068*	**0.6103**	0.0069	*0.6610*	**0.6634**	0.0083	*0.6904*	**0.6946**
		0.95	0.0117	**0.6874**	*0.6582*	0.0074	*0.7400*	**0.7406**	0.0082	*0.8128*	**0.8171**

Table 14: Average values of R^*_{C} (CCA), R^*_{N} (NCCA), and R^*_{P} (PNCCA) for (C)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
n	δ	ρ_x	R^*_{C}	R^*_{N}	R^*_{P}	R^*_{C}	R^*_{N}	R^*_{P}	R^*_{C}	R^*_{N}	R^*_{P}
30	1	0.5	0.0074	**0.2507**	*0.2479*	0.0047	*0.1690*	**0.2528**	0.0050	*0.1576*	**0.4187**
		0.8	0.0063	*0.3503*	**0.4134**	0.0016	*0.1914*	**0.5353**	0.0032	*0.2065*	**1.1178**
		0.95	0.0016	*0.5057*	**0.6864**	0.0012	*0.2715*	**1.4164**	0.0005	*0.1110*	**2.3199**
	3	0.5	0.0082	**0.6628**	*0.6571*	0.0069	*0.6211*	**0.7752**	0.0046	*0.8311*	**1.0931**
		0.8	0.0023	*1.1161*	**1.1373**	0.0028	*0.8099*	**1.4641**	0.0019	*0.9546*	**2.3910**
		0.95	0.0012	*1.4540*	**1.4771**	0.0011	*0.6674*	**2.5052**	0.0006	*0.3961*	**4.7958**
50	1	0.5	0.0064	**0.3006**	*0.2786*	0.0051	*0.2095*	**0.3011**	0.0045	*0.2809*	**0.4944**
		0.8	0.0049	*0.3943*	**0.5459**	0.0028	*0.4195*	**0.7251**	0.0018	*0.6204*	**1.2818**
		0.95	0.0012	*0.4479*	**0.8000**	0.0020	*0.6926*	**1.7492**	0.0005	*0.5374*	**2.7939**
	3	0.5	0.0067	**0.7499**	*0.7488*	0.0053	*0.8343*	**0.8708**	0.0027	*0.9939*	**1.0766**
		0.8	0.0029	*1.3193*	**1.3724**	0.0029	*1.5655*	**1.7197**	0.0019	*2.1370*	**2.4582**
		0.95	0.0009	*1.5079*	**1.7138**	0.0008	*2.2223*	**2.9143**	0.0004	*3.1285*	**5.4998**
100	1	0.5	0.0057	*0.3862*	**0.3951**	0.0053	*0.3193*	**0.3888**	0.0051	*0.3464*	**0.5038**
		0.8	0.0044	*0.5380*	**0.6380**	0.0015	*0.6137*	**0.8462**	0.0023	*0.7448*	**1.2510**
		0.95	0.0013	*0.5335*	**0.8878**	0.0007	*0.8215*	**1.8003**	0.0005	*0.8568*	**2.8302**
	3	0.5	0.0083	*0.8072*	**0.8198**	0.0062	*0.9455*	**0.9823**	0.0036	*1.0074*	**1.0798**
		0.8	0.0025	*1.3925*	**1.4396**	0.0029	*1.7927*	**1.8991**	0.0012	*2.1214*	**2.3359**
		0.95	0.0008	*1.6185*	**1.7614**	0.0006	*2.4415*	**3.0633**	0.0004	*3.8614*	**5.3736**

Table 15: Average values of R_{C} (CCA), R_{N} (NCCA), and R_{P} (PNCCA) for (C)

			$p_0 = 3$			$p_0 = 5$			$p_0 = 8$		
n	δ	ρ_x	R_{C}	R_{N}	R_{P}	R_{C}	R_{N}	R_{P}	R_{C}	R_{N}	R_{P}
30	1	0.5	0.0091	**0.2365**	*0.2303*	0.0068	*0.1632*	**0.1979**	0.0075	*0.1450*	**0.2500**
		0.8	0.0109	*0.2647*	**0.2782**	0.0044	*0.1614*	**0.2375**	0.0106	*0.2144*	**0.3470**
		0.95	0.0073	*0.3297*	**0.3328**	0.0082	*0.3483*	**0.4200**	0.0069	*0.3166*	**0.4382**
	3	0.5	0.0102	**0.6314**	*0.6229*	0.0080	*0.5651*	**0.6183**	0.0058	*0.5799*	**0.6492**
		0.8	0.0039	**0.7200**	*0.7124*	0.0075	*0.5646*	**0.6311**	0.0071	*0.6006*	**0.6918**
		0.95	0.0047	**0.6615**	*0.6501*	0.0064	*0.6208*	**0.6864**	0.0090	*0.7442*	**0.8442**
50	1	0.5	0.0080	**0.2652**	*0.2436*	0.0070	*0.1916*	**0.2275**	0.0072	*0.2284*	**0.3022**
		0.8	0.0099	*0.2826*	**0.3228**	0.0062	*0.2637*	**0.2889**	0.0066	*0.3398*	**0.3711**
		0.95	0.0081	*0.3453*	**0.3653**	0.0140	*0.4428*	**0.4556**	0.0074	*0.4569*	**0.4812**
	3	0.5	0.0101	**0.6612**	*0.6608*	0.0057	*0.6433*	**0.6539**	0.0045	*0.6553*	**0.6696**
		0.8	0.0051	*0.7472*	**0.7549**	0.0051	*0.6589*	**0.6675**	0.0086	*0.7003*	**0.7077**
		0.95	0.0094	*0.7242*	**0.7277**	0.0044	*0.7123*	**0.7169**	0.0071	*0.8695*	**0.8770**
100	1	0.5	0.0072	*0.3245*	**0.3248**	0.0068	*0.2473*	**0.2661**	0.0082	*0.2686*	**0.3179**
		0.8	0.0109	*0.3446*	**0.3615**	0.0045	*0.3084*	**0.3126**	0.0081	*0.3626*	**0.3810**
		0.95	0.0106	*0.3779*	**0.3827**	0.0072	*0.4697*	**0.4821**	0.0071	*0.4818*	**0.4942**
	3	0.5	0.0121	*0.6798*	**0.6822**	0.0065	*0.6525*	**0.6588**	0.0055	*0.6696*	**0.6825**
		0.8	0.0056	*0.7632*	**0.7666**	0.0083	*0.6697*	**0.6705**	0.0049	*0.7160*	**0.7202**
		0.95	0.0075	**0.7381**	*0.7337*	0.0059	*0.7343*	**0.7444**	0.0063	*0.8794*	**0.8823**