Tests for Two Mean Vectors and Simultaneous Confidence Intervals for Multiple Comparisons with Three-step Monotone Missing Data

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Abstract

In this paper, we consider the problem of testing the equality of two mean vectors when the two data sets have three-step or two-step monotone patterns of missing observations. Using the maximum likelihood estimators (MLEs) of the mean vectors and the covariance matrix for three-step monotone missing data, we obtain the MLEs and Hotelling's T^2-type statistics and their approximate upper percentiles in the case of data with different two-step monotone missing data patterns. Further, we consider multivariate multiple comparisons for mean vectors with three-step monotone missing data. Approximate simultaneous confidence intervals for comparisons with a control are obtained using Bonferroni's approximate upper percentiles of the $T^2_{\max \cdot \mathrm{c}}$ statistic. Finally, the accuracy of the approximations is investigated via Monte Carlo simulation.

Key Words and Phrases: Comparisons with a control; Maximum likelihood estimator; Pairwise comparisons; T^2-type statistic; Two-sample problem

1. Introduction

The two-sample problem of testing for mean vectors with three-step monotone missing data is considered in this paper. For the one-sample problem, the maximum likelihood estimators (MLEs) of the mean vector and the covariance matrix in the case of k-step monotone missing data under multivariate normality were given by Jinadasa and Tracy (1992). Further, Kanda and Fujikoshi (1998) discussed the distribution of the MLEs in the case of k-step monotone missing data. A simplified Hotelling's T^2-type statistic and its approximation to the upper percentiles in the case of three-step monotone missing data are given by Yagi and Seo (2014a) . Yagi and Seo (2014b) extends the one-sample problem in Yagi and Seo (2014a) to the two-sample or m-sample problem in the case of three-step monotone missing data. In this paper, we consider the problem of testing the equality of two mean vectors when the two data sets have different two-step monotone missing data patterns. In the case of data with same two-step monotone missing data patterns, Seko, Kawasaki and Seo (2011) derived a Hotelling's T^2-type statistic and the likelihood ratio test statistic and their approximate upper percentiles. In addition, as another approach, Yu, Krishnamoorthy and Pannala (2006) gave the approximate distribution of the Hotelling's T^2-type statistic. Recently, Seko (2012) discussed the tests for mean vectors with two-step monotone missing data for the m-sample problem. Under the two-sample problem, in this paper, we propose a simplified Hotelling's T^2-type statistic and its approximate upper percentile in the case of data with different two-step monotone missing data patterns. Further, we discuss the simultaneous confidence intervals when each data set has three-step monotone missing observations. Throughout this paper, we assume that the data are missing completely at random (MCAR).

The remainder of this paper is organized as follows. In Section 2, we present the MLEs of the mean vector and the covariance matrix for the m-sample problem that includes the two-sample problem when the data sets have three-step monotone missing data patterns. Further, the MLEs are obtained in the case of data with different two-step monotone missing data patterns. In Section 3, we present the Hotelling's T^2-type statistic to test the equality of two mean vectors and its approximate upper percentiles when two data sets have different two-step monotone missing data patterns. In Section 4, we present the simultaneous confidence intervals for comparisons with a control under the m-sample problem. In order to obtain the simultaneous confidence intervals, we derive approximate upper percentile of the $T^2_{\max\cdot\mathrm{c}}$ statistic via Bonferroni's approximation. Finally, in Section 5, we present some simulation results and state our conclusions.

2. Three-step and two-step monotone missing data and MLE

Let $\boldsymbol{x}_{i1}^{(\ell)}, \boldsymbol{x}_{i2}^{(\ell)}, \ldots, \boldsymbol{x}_{in_i^{(\ell)}}^{(\ell)}$ be distributed as $N_{p_i}(\boldsymbol{\mu}_i^{(\ell)}, \boldsymbol{\Sigma}_i)$ for $i = 1, 2, 3$ and $\ell = 1, 2, \ldots, m$, where $\boldsymbol{\mu}_i^{(\ell)} = (\mu_1^{(\ell)}, \mu_2^{(\ell)}, \ldots, \mu_{p_i}^{(\ell)})'$ and $\boldsymbol{\Sigma}_i$ is the $p_i \times p_i$ covariance matrix, where $p = p_1 > p_2 > p_3 > 0$ and $n_1^{(\ell)} > p$. That is, $(\boldsymbol{x}_{11}^{(\ell)}, \boldsymbol{x}_{12}^{(\ell)}, \ldots, \boldsymbol{x}_{1n_1^{(\ell)}}^{(\ell)})'$ is an $n_1^{(\ell)} \times p_1$ complete data set, and $(\boldsymbol{x}_{21}^{(\ell)}, \boldsymbol{x}_{22}^{(\ell)}, \ldots, \boldsymbol{x}_{2n_2^{(\ell)}}^{(\ell)})'$ and $(\boldsymbol{x}_{31}^{(\ell)}, \boldsymbol{x}_{32}^{(\ell)}, \ldots, \boldsymbol{x}_{3n_3^{(\ell)}}^{(\ell)})'$ are $n_2^{(\ell)} \times p_2$ and $n_3^{(\ell)} \times p_3$ incomplete data sets, respectively. For such a data sets, see Figure 1, where " $*$ " indicates a missing observation.

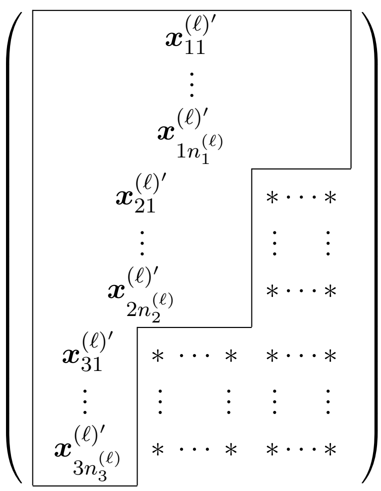

Figure 1: Three-step monotone missing data set

Further, let

$$\overline{\boldsymbol{x}}_i^{(\ell)} = \frac{1}{n_i^{(\ell)}} \sum_{j=1}^{n_i^{(\ell)}} \boldsymbol{x}_{ij}^{(\ell)},$$

$$\boldsymbol{E}_i^{(\ell)} = \sum_{j=1}^{n_i^{(\ell)}} (\boldsymbol{x}_{ij}^{(\ell)} - \overline{\boldsymbol{x}}_i^{(\ell)})(\boldsymbol{x}_{ij}^{(\ell)} - \overline{\boldsymbol{x}}_i^{(\ell)})', \ i = 1, 2, 3.$$

Then, the following theorem is given by Yagi and Seo (2014b). For some notations for the vector and the matrix, see Yagi and Seo (2014b).

Theorem 2.1. (Yagi and Seo (2014b)) *Let* $\boldsymbol{x}_{ij}^{(\ell)}$, $i = 1, 2, 3$, $j = 1, 2, \ldots, n_i^{(\ell)}$, $\ell = 1, 2, \ldots, m$ *be the* j*-th random vector of the* i*-th step from the* ℓ*-th population distributed as* $N_{p_i}(\boldsymbol{\mu}_i^{(\ell)}, \boldsymbol{\Sigma}_i)$*. Then, the MLEs of* $\boldsymbol{\mu}_i^{(\ell)}$, $\ell = 1, 2, \ldots, m$ *are given by*

$$\widehat{\boldsymbol{\mu}}^{(\ell)} = \overline{\boldsymbol{x}}_1^{(\ell)} + \widehat{\boldsymbol{T}}_2^{[pl]} \boldsymbol{d}_2^{(\ell)} + \widehat{\boldsymbol{T}}_2^{[pl]} \widehat{\boldsymbol{T}}_3^{[pl]} \boldsymbol{d}_3^{(\ell)},$$

where

$$\boldsymbol{d}_2^{(\ell)} = \frac{n_2^{(\ell)}}{N_3^{(\ell)}} \left[\overline{\boldsymbol{x}}_2^{(\ell)} - (\overline{\boldsymbol{x}}_1^{(\ell)})_2 \right], \ \boldsymbol{d}_3^{(\ell)} = \frac{n_3^{(\ell)}}{N_4^{(\ell)}} \left[\overline{\boldsymbol{x}}_3^{(\ell)} - \frac{1}{N_3^{(\ell)}} \left\{ n_1^{(\ell)} (\overline{\boldsymbol{x}}_1^{(\ell)})_3 + n_2^{(\ell)} (\overline{\boldsymbol{x}}_2^{(\ell)})_3 \right\} \right],$$

$$N_{i+1}^{(\ell)} = \sum_{j=1}^{i} n_j^{(\ell)}, \ \widehat{\boldsymbol{T}}_2^{[pl]} = \begin{pmatrix} \boldsymbol{I}_{p_2} \\ \widehat{\boldsymbol{\Sigma}}_{(1,2)}^{[pl]'} \widehat{\boldsymbol{\Sigma}}_2^{[pl]^{-1}} \end{pmatrix}, \ \widehat{\boldsymbol{T}}_3^{[pl]} = \begin{pmatrix} \boldsymbol{I}_{p_3} \\ \widehat{\boldsymbol{\Sigma}}_{(2,2)}^{[pl]'} \widehat{\boldsymbol{\Sigma}}_3^{[pl]^{-1}} \end{pmatrix},$$

and then, the MLE of $\boldsymbol{\Sigma}$ is given by

$$\begin{aligned} \widehat{\boldsymbol{\Sigma}}^{[pl]} &= \frac{1}{M_2} \sum_{\ell=1}^{m} \boldsymbol{E}_1^{(\ell)} \\ &+ \frac{1}{M_3} \left[\sum_{\ell=1}^{m} \boldsymbol{G}_2^{(\ell)} \left\{ \boldsymbol{E}_2^{(\ell)} + \frac{N_2^{(\ell)} N_3^{(\ell)}}{n_2^{(\ell)}} \boldsymbol{d}_2^{(\ell)} \boldsymbol{d}_2^{(\ell)'} - \frac{n_2^{(\ell)}}{N_2^{(\ell)}} \boldsymbol{L}_{11}^{(\ell)} \right\} \boldsymbol{G}_2^{(\ell)'} \right] \\ &+ \frac{1}{M_4} \left[\sum_{\ell=1}^{m} \boldsymbol{G}_2^{(\ell)} \boldsymbol{G}_3^{(\ell)} \left\{ \boldsymbol{E}_3^{(\ell)} + \frac{N_3^{(\ell)} N_4^{(\ell)}}{n_3^{(\ell)}} \boldsymbol{d}_3^{(\ell)} \boldsymbol{d}_3^{(\ell)'} - \frac{n_3^{(\ell)}}{N_3^{(\ell)}} \boldsymbol{L}_{21}^{(\ell)} \right\} \boldsymbol{G}_3^{(\ell)'} \boldsymbol{G}_2^{(\ell)'} \right], \end{aligned}$$

where

$$M_r = \sum_{\ell=1}^{m} N_r^{(\ell)}, \ r = 2, 3, 4,$$

$$\boldsymbol{G}_2^{(\ell)} = \begin{pmatrix} \boldsymbol{I}_{p_2} \\ \boldsymbol{L}_{12}^{(\ell)'} \boldsymbol{L}_{11}^{(\ell)^{-1}} \end{pmatrix}, \ \boldsymbol{G}_3^{(\ell)} = \begin{pmatrix} \boldsymbol{I}_{p_3} \\ \boldsymbol{L}_{22}^{(\ell)'} \boldsymbol{L}_{21}^{(\ell)^{-1}} \end{pmatrix},$$

$$\boldsymbol{L}_1^{(\ell)} = \boldsymbol{E}_1^{(\ell)},\ \boldsymbol{L}_2^{(\ell)} = \boldsymbol{L}_{11}^{(\ell)} + \boldsymbol{E}_2^{(\ell)} + \frac{N_2^{(\ell)} N_3^{(\ell)}}{n_2^{(\ell)}} \boldsymbol{d}_2^{(\ell)} \boldsymbol{d}_2^{(\ell)\prime},$$

$$\boldsymbol{L}_i^{(\ell)} = \begin{pmatrix} \boldsymbol{L}_{i1}^{(\ell)} & \boldsymbol{L}_{i2}^{(\ell)} \\ \boldsymbol{L}_{i2}^{(\ell)\prime} & \boldsymbol{L}_{i3}^{(\ell)} \end{pmatrix},\ i = 1, 2.$$

Using the results of Theorem 2.1, the Hotelling's T^2-type statistic to test for the equality of two mean vectors is given by Yagi and Seo (2014b). In this paper, we propose the Hotelling's T^2-type statistic when two data sets have different two-step monotone missing data patterns. That is, two data sets Π_1 and Π_2 are of the forms given in Figure 2.

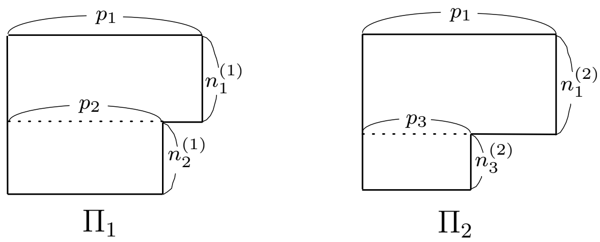

Figure 2: Different two-step monotone missing data patterns

Then, we can reduce Theorem 2.1 to the following result.

Theorem 2.2. *Let* $\boldsymbol{x}_{ij}^{(\ell)}$, $i = 1, 2, 3$, $j = 1, 2, \ldots, n_i^{(\ell)}$, $\ell = 1, 2$ *be the* j*-th random vector of the* i*-th step from the* ℓ*-th population distributed as* $N_{p_i}(\boldsymbol{\mu}_i^{(\ell)}, \boldsymbol{\Sigma}_i)$*, where* $n_3^{(1)} = 0$, $n_2^{(2)} = 0$*. Then, the MLEs of* $\boldsymbol{\mu}_i^{(\ell)}$, $\ell = 1, 2$ *and* $\boldsymbol{\Sigma}$ *are given by*

$$\widehat{\boldsymbol{\mu}}^{(1)} = \overline{\boldsymbol{x}}_1^{(1)} + \widehat{\boldsymbol{T}}_2^{[pl]} \boldsymbol{d}_2^{(1)},$$

$$\widehat{\boldsymbol{\mu}}^{(2)} = \overline{\boldsymbol{x}}_1^{(2)} + \widehat{\boldsymbol{T}}_2^{[pl]} \widehat{\boldsymbol{T}}_3^{[pl]} \boldsymbol{d}_3^{(2)},$$

$$\begin{aligned}
\widehat{\boldsymbol{\Sigma}}^{[pl]} &= \frac{1}{N_2^{(1)} + N_2^{(2)}} \sum_{\ell=1}^{2} \boldsymbol{E}_1^{(\ell)} \\
&\quad + \frac{1}{N_3^{(1)} + N_3^{(2)}} \left[\boldsymbol{G}_2^{(1)} \left\{ \boldsymbol{E}_2^{(1)} + \frac{N_2^{(1)} N_3^{(1)}}{n_2^{(1)}} \boldsymbol{d}_2^{(1)} \boldsymbol{d}_2^{(1)\prime} - \frac{n_2^{(1)}}{N_2^{(1)}} \boldsymbol{L}_{11}^{(1)} \right\} \boldsymbol{G}_2^{(1)\prime} \right] \\
&\quad + \frac{1}{N_4^{(1)} + N_4^{(2)}} \left[\boldsymbol{G}_2^{(2)} \boldsymbol{G}_3^{(2)} \left\{ \boldsymbol{E}_3^{(2)} + \frac{N_3^{(2)} N_4^{(2)}}{n_3^{(2)}} \boldsymbol{d}_3^{(2)} \boldsymbol{d}_3^{(2)\prime} - \frac{n_3^{(2)}}{N_3^{(2)}} \boldsymbol{L}_{21}^{(2)} \right\} \boldsymbol{G}_3^{(2)\prime} \boldsymbol{G}_2^{(2)\prime} \right],
\end{aligned}$$

where

$$N_4^{(1)} = n_1^{(1)} + n_2^{(1)} (= N_3^{(1)}),$$
$$N_3^{(2)} = n_1^{(2)} (= N_2^{(2)}),\ N_4^{(2)} = n_1^{(2)} + n_3^{(2)}.$$

3. Hotelling's T^2-type statistics

In this section, we first consider the hypothesis test $H_0 : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)}$ vs. $H_1 : \boldsymbol{\mu}^{(1)} \neq \boldsymbol{\mu}^{(2)}$ when the data have a three-step monotone missing data pattern. In this case, the Hotelling's T^2-type statistic is given by

$$\widetilde{T}^2 = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})' \, \widetilde{\boldsymbol{\Gamma}}^{[pl]^{-1}} (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}),$$

where $\widehat{\boldsymbol{\mu}}^{(\ell)} = \overline{\boldsymbol{x}}_1^{(\ell)} + \widehat{\boldsymbol{T}}_2^{[pl]} \boldsymbol{d}_2^{(\ell)} + \widehat{\boldsymbol{T}}_2^{[pl]} \widehat{\boldsymbol{T}}_3^{[pl]} \boldsymbol{d}_3^{(\ell)},\ \ell = 1, 2,$ and $\widetilde{\boldsymbol{\Gamma}}^{[pl]}$ is an estimator of $\boldsymbol{\Gamma} = \mathrm{Cov}[\widetilde{\boldsymbol{\mu}}^{(1)} - \widetilde{\boldsymbol{\mu}}^{(2)}]$, $\widetilde{\boldsymbol{\mu}}^{(\ell)} = \overline{\boldsymbol{x}}_1^{(\ell)} + \boldsymbol{T}_2 \boldsymbol{d}_2^{(\ell)} + \boldsymbol{T}_2 \boldsymbol{T}_3 \boldsymbol{d}_3^{(\ell)},\ \ell = 1, 2$. Then, we have

$$\widetilde{\boldsymbol{\Gamma}}^{[pl]} = \left(\sum_{\ell=1}^{2} \frac{1}{N_2^{(\ell)}} \right) \widehat{\boldsymbol{\Sigma}}_1^{[pl]} - \left(\sum_{\ell=1}^{2} \frac{n_2^{(\ell)}}{N_2^{(\ell)} N_3^{(\ell)}} \right) \widehat{\boldsymbol{U}}_2^{[pl]} - \left(\sum_{\ell=1}^{2} \frac{n_3^{(\ell)}}{N_3^{(\ell)} N_4^{(\ell)}} \right) \widehat{\boldsymbol{U}}_3^{[pl]},$$

where $\widehat{\boldsymbol{\Sigma}}^{[pl]}$ is the MLE for $m = 2$ in Theorem 2.1,

$$\widehat{\boldsymbol{U}}_2^{[pl]} = \begin{pmatrix} \widehat{\boldsymbol{\Sigma}}_2^{[pl]} \\ \widehat{\boldsymbol{\Sigma}}_{22}^{[pl]'} \end{pmatrix} \widehat{\boldsymbol{T}}_2^{[pl]'},\ \widehat{\boldsymbol{U}}_3^{[pl]} = \begin{pmatrix} \widehat{\boldsymbol{\Sigma}}_3^{[pl]} \\ \widehat{\boldsymbol{\Sigma}}_{32}^{[pl]'} \end{pmatrix} \widehat{\boldsymbol{T}}_3^{[pl]'} \widehat{\boldsymbol{T}}_2^{[pl]'}.$$

We note that this result of $\widetilde{\boldsymbol{\Gamma}}^{[pl]}$ is a correction of the result in Yagi and Seo (2014b).

Further, if the two data sets have different two-step monotone missing data patterns, then the Hotelling's T^2-type statistic is reduced to

$$\widetilde{T}^2 = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})' \, \widetilde{\boldsymbol{\Gamma}}^{[pl]^{-1}} (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}),$$

where $\widetilde{\boldsymbol{\mu}}^{(1)} = \overline{\boldsymbol{x}}_1^{(1)} + \boldsymbol{T}_2 \boldsymbol{d}_2^{(1)},\ \ \widehat{\boldsymbol{\mu}}^{(2)} = \overline{\boldsymbol{x}}_1^{(2)} + \boldsymbol{T}_2 \boldsymbol{T}_3 \boldsymbol{d}_3^{(2)}$, and

$$\widetilde{\boldsymbol{\Gamma}}^{[pl]} = \left(\sum_{\ell=1}^{2} \frac{1}{N_2^{(\ell)}} \right) \widehat{\boldsymbol{\Sigma}}_1^{[pl]} - \frac{n_2^{(1)}}{N_2^{(1)} N_3^{(1)}} \widehat{\boldsymbol{U}}_2^{[pl]} - \frac{n_3^{(2)}}{N_3^{(2)} N_4^{(2)}} \widehat{\boldsymbol{U}}_3^{[pl]}$$

where

$$N_3^{(2)} = n_1^{(2)},\ N_4^{(2)} = n_1^{(2)} + n_3^{(2)},$$
$$\widehat{\boldsymbol{U}}_2^{[pl]} = \begin{pmatrix} \widehat{\boldsymbol{\Sigma}}_2^{[pl]} \\ \widehat{\boldsymbol{\Sigma}}_{22}^{[pl]\prime} \end{pmatrix} \widehat{\boldsymbol{T}}_2^{[pl]\prime},\ \widehat{\boldsymbol{U}}_3^{[pl]} = \begin{pmatrix} \widehat{\boldsymbol{\Sigma}}_3^{[pl]} \\ \widehat{\boldsymbol{\Sigma}}_{32}^{[pl]\prime} \end{pmatrix} \widehat{\boldsymbol{T}}_3^{[pl]\prime} \widehat{\boldsymbol{T}}_2^{[pl]\prime},$$

and $\widehat{\boldsymbol{\Sigma}}^{[pl]}$ is the MLE in Theorem 2.2. Indeed, Kanda and Fujikoshi (1998) obtained $\mathrm{Cov}[\widehat{\boldsymbol{\mu}}]$ with two-step monotone missing data under the one-sample problem. Under the two-sample problem, we can easily obtain $\mathrm{Cov}[\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}]$ with different two-step monotone missing data: therefore we can obtain T^2 statistic as follows,

$$T^2 = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})' \, \widehat{\boldsymbol{\Gamma}}^{[pl]^{-1}} (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}),$$

where

$$\widehat{\boldsymbol{\mu}}^{(1)} = \overline{\boldsymbol{x}}_1^{(1)} + \widehat{\boldsymbol{T}}_2^{[pl]} \boldsymbol{d}_2^{(1)},\ \widehat{\boldsymbol{\mu}}^{(2)} = \overline{\boldsymbol{x}}_1^{(2)} + \widehat{\boldsymbol{T}}_2^{[pl]} \widehat{\boldsymbol{T}}_3^{[pl]} \boldsymbol{d}_3^{(2)},$$
$$\widehat{\boldsymbol{\Gamma}}^{[pl]} = \sum_{\ell=1}^{2} (\widehat{\mathrm{Cov}}[\widehat{\boldsymbol{\mu}}^{(\ell)}] + \widehat{\boldsymbol{R}}_{(\ell)}^{[pl]}),$$
$$\widehat{\boldsymbol{R}}_{(1)}^{[pl]} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \dfrac{n_2^{(1)} p_2}{N_2^{(1)} N_3^{(1)} (N_2^{(1)} - p_2 - 2)} \widehat{\boldsymbol{\Sigma}}_{23\cdot 2}^{[pl]} \end{pmatrix},\ \widehat{\boldsymbol{\Sigma}}_{23\cdot 2}^{[pl]} = \widehat{\boldsymbol{\Sigma}}_{23}^{[pl]} - \widehat{\boldsymbol{\Sigma}}_{22}^{[pl]\prime} (\widehat{\boldsymbol{\Sigma}}_2^{[pl]})^{-1} \widehat{\boldsymbol{\Sigma}}_{22}^{[pl]},$$
$$\widehat{\boldsymbol{R}}_{(2)}^{[pl]} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \dfrac{n_3^{(2)} p_3}{N_2^{(2)} N_4^{(2)} (N_2^{(2)} - p_3 - 2)} \widehat{\boldsymbol{\Sigma}}_{33\cdot 3}^{[pl]} \end{pmatrix},\ \widehat{\boldsymbol{\Sigma}}_{33\cdot 3}^{[pl]} = \widehat{\boldsymbol{\Sigma}}_{33}^{[pl]} - \widehat{\boldsymbol{\Sigma}}_{32}^{[pl]\prime} (\widehat{\boldsymbol{\Sigma}}_3^{[pl]})^{-1} \widehat{\boldsymbol{\Sigma}}_{32}^{[pl]},$$
$$N_2^{(\ell)} > p_j + 2,\ j = 2, 3.$$

For details on the case in which two data sets have the same two-step monotone missing data pattern, see Yagi and Seo (2015). Further, as two approximate values to the upper percentiles of the T^2 or $\widetilde{T}^2$ statistic, we can obtain the following theorem.

Theorem 3.1. *Suppose that two data sets have different two-step monotone missing data patterns. Then, the two approximate upper 100α percentiles of the* T^2 *(or* $\widetilde{T}^2$*)*

statistic are given by

$$t^2_{\mathrm{YS\cdot L}}(\alpha) = (1-d)T^2_{n,\alpha} + dT^2_{N,\alpha},$$
$$t^2_{\mathrm{YS}\cdot F}(\alpha) = \frac{n^* p_1}{n^* - p_1 - 1} F_{p_1, n^*-p_1-1,\alpha},$$

where

$$d = \frac{n_2^{(1)} p_2 + n_3^{(2)} p_3}{\left(n_2^{(1)} + n_3^{(2)}\right) p_1}, \ T^2_{n,\alpha} = \frac{n p_1}{n - p_1 - 1} F_{p_1, n-p_1-1,\alpha},$$
$$T^2_{N,\alpha} = \frac{N p_1}{N - p_1 - 1} F_{p_1, N-p_1-1,\alpha}, \ n = n_1^{(1)} + n_1^{(2)},$$
$$N = n_1^{(1)} + n_2^{(1)} + n_1^{(2)} + n_3^{(2)}, \ n^* = \frac{1}{p_1}\left(p_1 \sum_{\ell=1}^{2} n_1^{(\ell)} + n_2^{(1)} p_2 + n_3^{(2)} p_3\right).$$

Next, under the two-sample problem, we can consider the simultaneous confidence intervals when each data set has three-step monotone missing observations.

For any nonnull vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_p)'$, the simultaneous confidence intervals for $\boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)})$ with the confidence level $(1-\alpha)$ are given by

$$\boldsymbol{a}'(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}) - \sqrt{L} \le \boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}) \le \boldsymbol{a}'(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}) + \sqrt{L}, \ \forall \boldsymbol{a} \in \boldsymbol{R}^p - \{\boldsymbol{0}\},$$

where $L = t^2(\alpha)\boldsymbol{a}'\widehat{\boldsymbol{\Gamma}}^{[pl]}\boldsymbol{a}$, $t^2(\alpha)$ is the upper 100α percentile of the $T^2 (= (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})'\widehat{\boldsymbol{\Gamma}}^{[pl]^{-1}}(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}))$ statistic, and $\widehat{\boldsymbol{\Gamma}}^{[pl]}$ is an estimator of $\mathrm{Cov}[\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}]$. However, it is not easy to obtain $t^2(\alpha)$. Therefore, using the approximate upper percentiles of the $\widetilde{T}^2$ statistic, $t^2_{\mathrm{YS\cdot L}}(\alpha)$ or $t^2_{\mathrm{YS}\cdot F}(\alpha)$, for any nonnull vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_p)'$, the approximate simultaneous confidence intervals for $\boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)})$ are given by

$$\boldsymbol{a}'(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}) - \sqrt{L_{\mathrm{app}}} \le \boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}) \le \boldsymbol{a}'(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}) + \sqrt{L_{\mathrm{app}}}, \ \forall \boldsymbol{a} \in \boldsymbol{R}^p - \{\boldsymbol{0}\},$$

where $L_{\mathrm{app}} = t^2_{\mathrm{app}}(\alpha)(\boldsymbol{a}'\widetilde{\boldsymbol{\Gamma}}^{[pl]}\boldsymbol{a})$ and the value of $t^2_{\mathrm{app}}(\alpha)$ is $t^2_{\mathrm{YS\cdot L}}(\alpha)$ or $t^2_{\mathrm{YS}\cdot F}(\alpha)$ in Theorem 3.1.

4. Simultaneous confidence intervals for multiple comparisons among mean vectors

Under the m-sample case, we consider the simultaneous confidence intervals for comparisons with a control when each data set has three-step monotone missing observations. For the simultaneous confidence intervals for pairwise comparisons among mean vectors, see Yagi and Seo (2014b).

Let $\boldsymbol{x}_{i1}^{(\ell)}, \boldsymbol{x}_{i2}^{(\ell)}, \ldots, \boldsymbol{x}_{in_i^{(\ell)}}^{(\ell)}$ be distributed as $N_{p_i}(\boldsymbol{\mu}_i^{(\ell)}, \boldsymbol{\Sigma}_i)$ for $i = 1, 2, 3$ and $\ell = 1, 2, \ldots, m$. For the case of the comparisons with a control, let $\boldsymbol{\mu}^{(1)}$ be a control and define the $T^2_{\max\cdot\mathrm{c}}$ statistic as

$$T^2_{\max\cdot\mathrm{c}} = \max_{2 \leq b \leq m} T^2_{1b},$$

where $T^2_{1b} = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(a)})' \widehat{\boldsymbol{\Gamma}}_{1b}^{[pl]^{-1}} (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(b)})$ and $\widehat{\boldsymbol{\Gamma}}_{1b}^{[pl]}$ is an estimator of $\mathrm{Cov}[\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(b)}]$. Then, the simultaneous confidence intervals for $\boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(b)})$, $2 \leq b \leq m$ are given by

$$\boldsymbol{a}'(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(b)}) - \sqrt{L_\mathrm{c}} \leq \boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(b)}) \leq \boldsymbol{a}'(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(b)}) + \sqrt{L_\mathrm{c}},$$
$$2 \leq b \leq m,\ \forall \boldsymbol{a} \in \boldsymbol{R}^p - \{\boldsymbol{0}\},$$

where $L_\mathrm{c} = t^2_{\max\cdot\mathrm{c}}(\alpha)(\boldsymbol{a}'\widehat{\boldsymbol{\Gamma}}_{1b}^{[pl]}\boldsymbol{a})$ and $t^2_{\max\cdot\mathrm{c}}(\alpha)$ is the upper percentile of the $T^2_{\max\cdot\mathrm{c}}$ statistic. However, it is not easy to obtain $t^2_{\max\cdot\mathrm{c}}(\alpha)$ even under non-missing multivariate normality (see Seo and Siotani (1992), Seo, Mano and Fujikoshi (1994)). Therefore, in this paper, we adopt Bonferroni's approximation, which is one of the solutions to this problem. Let $n_i^{(1)} = n_i^{(2)} = \cdots = n_i^{(m)}$, $i = 1, 2, 3$: then, the null distributions of T^2_{1b} are identical. Therefore, the approximate simultaneous confidence intervals for comparisons with a control are given by

$$\boldsymbol{a}'(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(b)}) - \sqrt{L^*_\mathrm{c}} \leq \boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(b)}) \leq \boldsymbol{a}'(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(b)}) + \sqrt{L^*_\mathrm{c}},$$
$$2 \leq b \leq m,\ \forall \boldsymbol{a} \in \boldsymbol{R}^p - \{\boldsymbol{0}\},$$

where $L^*_\mathrm{c} = t^2_{\mathrm{Bon}}(\alpha_\mathrm{c})(\boldsymbol{a}'\widetilde{\boldsymbol{\Gamma}}_{1b}^{[pl]}\boldsymbol{a})$, the value of $t^2_{\mathrm{Bon}}(\alpha_\mathrm{c})$ is $t^2_{\mathrm{YS\cdot L}}(\alpha_\mathrm{c})$ or $t^2_{\mathrm{YS}\cdot F}(\alpha_\mathrm{c})$ and $\alpha_\mathrm{c} = \alpha/(m-1)$. We note that $\widetilde{\boldsymbol{\Gamma}}_{1b}^{[pl]}$ is an estimator of $\mathrm{Cov}[\widetilde{\boldsymbol{\mu}}^{(1)} - \widetilde{\boldsymbol{\mu}}^{(b)}]$ and is estimated by the use of $\widehat{\mathbf{\Sigma}}^{[pl]}$ in Theorem 2.1. Also, $t^2_{\mathrm{YS}\cdot \mathrm{L}}(\alpha_{\mathrm{c}})$ and $t^2_{\mathrm{YS}\cdot F}(\alpha_{\mathrm{c}})$ are given by

$$t^2_{\mathrm{YS}\cdot \mathrm{L}}(\alpha_{\mathrm{c}}) = (1-d)T^2_{n,\alpha_{\mathrm{c}}} + dT^2_{N,\alpha_{\mathrm{c}}},$$

$$t^2_{\mathrm{YS}\cdot F}(\alpha_{\mathrm{c}}) = \frac{n^* p_1}{n^* - p_1 - 1} F_{p_1, n^* - p_1 - 1, \alpha_{\mathrm{c}}},$$

where

$$d = \frac{\sum_{\ell=1}^{2}\sum_{i=2}^{3} n_i^{(\ell)} p_i}{p_1 \sum_{\ell=1}^{2}\sum_{i=2}^{3} n_i^{(\ell)}},$$

$$T^2_{n,\alpha_{\mathrm{c}}} = \frac{np_1}{n-p_1-1} F_{p_1, n-p_1-1, \alpha_{\mathrm{c}}},\ T^2_{N,\alpha_{\mathrm{c}}} = \frac{Np_1}{N-p_1-1} F_{p_1, N-p_1-1, \alpha_{\mathrm{c}}},$$

$$n = n_1^{(1)} + n_1^{(2)},\ N = N_4^{(1)} + N_4^{(2)},\ n^* = \frac{1}{p_1}\sum_{\ell=1}^{2}\sum_{i=1}^{3} n_i^{(\ell)} p_i$$

and $F_{p,q,\alpha}$ is the upper 100α percentile of the F distribution with p and q degrees of freedom.

5. Simulation studies

In order to investigate the accuracy of some of the approximations, we compute the upper percentiles of the T^2, $\widetilde{T}^2$ and $\widetilde{T}^2_{\mathrm{max}\cdot \mathrm{c}}$ statistics via Monte Carlo simulation. For each parameter, the simulation was involved for 1,000,000 trials based on three-step and two-step monotone missing data sets.

The simulation results related to the upper percentiles of the $\widetilde{T}^2$ statistic and their approximations in the case of two-step monotone missing data are summarized in Tables 1 $\sim$ 6. Computations are carried out for the following three cases:

$$\text{Case I} \ : (p_1, p_2, p_1, p_3) = (6,4,6,2), (6,5,6,1), (12,8,12,4), (12,10,12,2),$$

$$n_1^{(1)} = n_1^{(2)} = 20, 30, 50, 100, 200, 300,$$

$$(n_2^{(1)}, n_3^{(2)}) = (10,10), (10,20), (20,10), (20,20), \ \alpha = 0.05, 0.01,$$

$$\text{Case II} \ : (p_1, p_2) = (6,3), (12,6),$$

$$n_1^{(1)} = n_1^{(2)} = 20, 30, 50, 100, 200, 300,$$

$$(n_2^{(1)}, n_2^{(2)}) = (10,10), (10,20), (20,20), \ \alpha = 0.05, 0.01.$$

Tables 1 $\sim$ 4 present the results for the unbalanced case of the dimensions. Tables 1 $\sim$ 6 list the simulated upper 100α percentiles of the T^2 and $\widetilde{T}^2$ statistics ($t^2_{\text{simu}}(\alpha)$, $\widetilde{t}^2_{\text{simu}}(\alpha)$), the approximate upper 100α percentiles of T^2 and $\widetilde{T}^2$ ($t^2_{\text{YS}\cdot\text{L}}(\alpha)$, $t^2_{\text{YS}\cdot F}(\alpha)$), and the upper 100α percentiles of the χ^2 distribution with p degrees of freedom ($\chi^2_{p,\alpha}$). In the tables, we denote that $t^2_{\text{simu}}(\alpha)$, $\widetilde{t}^2_{\text{simu}}(\alpha)$, $t^2_{\text{YS}\cdot\text{L}}(\alpha)$, and $t^2_{\text{YS}\cdot F}(\alpha)$ as t^2_{simu}, $\widetilde{t}^2_{\text{simu}}$, $t^2_{\text{YS}\cdot\text{L}}$, and $t^2_{\text{YS}\cdot F}$ respectively. In addition, we provide the simulated coverage probabilities for the approximate upper 100α percentiles given by

$$\text{CP}(t^2_{\text{YS}\cdot\text{L}}(\alpha)) = 1 - \Pr\{T^2 > t^2_{\text{YS}\cdot\text{L}}(\alpha)\},$$

$$\text{CP}(t^2_{\text{YS}\cdot F}(\alpha)) = 1 - \Pr\{T^2 > t^2_{\text{YS}\cdot F}(\alpha)\},$$

$$\text{CP}(\chi^2_{p,\alpha}) = 1 - \Pr\{T^2 > \chi^2_{p,\alpha}\},$$

$$\widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha)) = 1 - \Pr\{\widetilde{T}^2 > t^2_{\text{YS}\cdot\text{L}}(\alpha)\},$$

$$\widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha)) = 1 - \Pr\{\widetilde{T}^2 > t^2_{\text{YS}\cdot F}(\alpha)\},$$

$$\widetilde{\text{CP}}(\chi^2_{p,\alpha}) = 1 - \Pr\{\widetilde{T}^2 > \chi^2_{p,\alpha}\}.$$

It may be noted from Tables 1 $\sim$ 6 that the simulated values are not close to the upper percentiles of the χ^2 distribution even when the sample size $n_1^{(\ell)}$ is moderately large. However, it is seen that the proposed approximations are accurate even for cases where $n_1^{(\ell)}$ is not large. In particular, it is noted that the values of $t^2_{\text{YS}\cdot\text{L}}(\alpha)$

are highly accurate for almost all cases. In other words, the simulated coverage probabilities for $t^2_{\mathrm{YS\cdot L}}(\alpha)$ are considerably close to the nominal level $1-\alpha$. Therefore, it can be concluded that the approximation $t^2_{\mathrm{YS\cdot L}}(\alpha)$ is highly accurate even for small samples and unbalanced cases when the data have a two-step monotone pattern of missing observations.

Next, in order to compare the approximate values with the simulated value in the cases of comparisons with a control, computations are carried out for the following case:

$$
\begin{aligned}
\text{Case III} \ :\ & m = 3, 6, 10,\ (p_1, p_2, p_3) = (6, 4, 2), (12, 8, 4),\ \alpha = 0.05, 0.01, \\
& \text{(i)}\ n_1^{(\ell)} = 13, 15, 20, 25,\ (n_2^{(\ell)}, n_3^{(\ell)}) = (5, 5), (10, 10), \\
& \text{(ii)}\ n_1^{(\ell)} = 30, 50, 100, 200,\ (n_2^{(\ell)}, n_3^{(\ell)}) = (10, 10), (20, 20).
\end{aligned}
$$

For the cases of pairwise comparisons, see Yagi and Seo (2014b).

Tables 7 $\sim$ 12 list the simulated upper 100α percentiles of the $\widetilde{T}^2_{\mathrm{max\cdot c}}$ statistic ($\widetilde{t}^2_{\mathrm{simu\cdot c}}(\alpha)$), the simulated upper $100\alpha_1$ percentiles of the $\widetilde{T}^2_{1b}$ statistic ($t^2_{\mathrm{simu\cdot Bon}}(\alpha_{\mathrm{c}})$) (that is, Bonferroni's approximation by Monte Carlo simulation), the approximate upper $100\alpha_{\mathrm{c}}$ percentiles of the $\widetilde{T}^2_{1b}$ statistic ($\widetilde{t}^2_{\mathrm{YS\cdot L}}(\alpha_{\mathrm{c}})$, $\widetilde{t}^2_{\mathrm{YS}\cdot F}(\alpha_{\mathrm{c}})$) and the upper $100\alpha_{\mathrm{c}}$ percentiles of the χ^2 distribution with p degrees of freedom ($\chi^2_{p,\alpha_{\mathrm{c}}}$). In the tables, we denote that $\widetilde{t}^2_{\mathrm{simu\cdot c}}(\alpha)$, $\widetilde{t}^2_{\mathrm{simu\cdot Bon}}(\alpha_{\mathrm{c}})$, $\widetilde{t}^2_{\mathrm{YS\cdot L}}(\alpha_{\mathrm{c}})$ and $\widetilde{t}^2_{\mathrm{YS}\cdot F}(\alpha_{\mathrm{c}})$ as $\widetilde{t}^2_{\mathrm{simu\cdot c}}$, $\widetilde{t}^2_{\mathrm{simu\cdot Bon}}$, $\widetilde{t}^2_{\mathrm{YS\cdot L}}$, and $\widetilde{t}^2_{\mathrm{YS}\cdot F}$, respectively. In addition, we provide the simulated coverage probabilities given by

$$\mathrm{CP}(t^2_{\mathrm{YS\cdot L}}(\alpha_{\mathrm{c}})) = 1 - \Pr\{\widetilde{T}^2_{1b} > t^2_{\mathrm{YS\cdot L}}(\alpha_{\mathrm{c}})\},$$

$$\mathrm{CP}(t^2_{\mathrm{YS}\cdot F}(\alpha_{\mathrm{c}})) = 1 - \Pr\{\widetilde{T}^2_{1b} > t^2_{\mathrm{YS}\cdot F}(\alpha_{\mathrm{c}})\},$$

$$\mathrm{CP}(\chi^2_{p,\alpha_{\mathrm{c}}}) = 1 - \Pr\{\widetilde{T}^2_{1b} > \chi^2_{p,\alpha_{\mathrm{c}}}\}.$$

It may be noted from Tables 7 $\sim$ 12 that the simulated values for $\widetilde{t}^2_{\mathrm{simu\cdot Bon}}(\alpha_{\mathrm{c}})$ are larger than the simulated values for $\widetilde{t}^2_{\mathrm{simu\cdot c}}(\alpha)$ because Bonferroni's approximation is always an overestimate for the $T^2_{\max}$-type statistic: this can be shown theoretically. It may be seen from the tables that the simulated values of $\widetilde{t}^2_{\mathrm{YS\cdot L}}(\alpha_{\mathrm{c}})$ and $\widetilde{t}^2_{\mathrm{YS}\cdot F}(\alpha_{\mathrm{c}})$ are closer to the simulated values $\widetilde{t}^2_{\mathrm{simu\cdot c}}(\alpha)$ when the sample size becomes large. Tables 10 and 12 give the simulation results and their approximations for cases where m and (p_1, p_2, p_3) are large. In these cases, it may be noted that the approximations are too conservative. The simulation studies show that $\widetilde{t}^2_{\mathrm{YS\cdot L}}(\alpha_{\mathrm{c}})$ is close to $\widetilde{t}^2_{\mathrm{simu\cdot c}}(\alpha)$ and is an accurate approximation in the cases where the sample size is large.

In conclusion, we developed the approximate upper percentiles of the Hotelling's T^2-type statistic for testing the equality of mean vectors and simultaneous confidence intervals with two-step or three-step monotone missing data. We can easily calculate the proposed approximate values and the accuracy of the approximations is considerably higher than that of the χ^2 approximations in almost all cases.

Table 1: Simulated and approximate values and coverage probabilities for $(p_1, p_2, p_1, p_3) = (6, 4, 6, 2)$ in Case I

Sample Size				Upper Percentile				Coverage Probability					
$n_1^{(1)}$	$n_2^{(1)}$	$n_1^{(2)}$	$n_3^{(2)}$	t^2_{simu}	$\widetilde{t}^2_{\text{simu}}$	$t^2_{\text{YS}\cdot\text{L}}$	$t^2_{\text{YS}\cdot F}$	$\text{CP}_{\text{YS}\cdot\text{L}}$	$\text{CP}_{\text{YS}\cdot F}$	CP_{χ^2}	$\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}}$	$\widetilde{\text{CP}}_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\chi^2}$
						$\alpha = 0.05$							
20	10	20	10	16.09	16.66	16.42	16.18	0.954	0.951	0.882	0.947	0.944	0.869
30	10	30	10	14.91	15.13	15.04	14.98	0.952	0.951	0.905	0.949	0.948	0.900
50	10	50	10	14.00	14.08	14.03	14.02	0.950	0.950	0.923	0.949	0.949	0.921
100	10	100	10	13.28	13.30	13.31	13.30	0.950	0.950	0.937	0.950	0.950	0.936
200	10	200	10	12.95	12.96	12.95	12.95	0.950	0.950	0.943	0.950	0.950	0.943
300	10	300	10	12.86	12.86	12.83	12.83	0.949	0.949	0.945	0.949	0.949	0.945
20	10	20	20	15.83	16.53	16.31	15.90	0.956	0.951	0.887	0.947	0.942	0.872
30	10	30	20	14.80	15.09	14.98	14.85	0.952	0.951	0.907	0.948	0.947	0.901
50	10	50	20	13.95	14.05	14.00	13.97	0.951	0.950	0.924	0.949	0.949	0.922
100	10	100	20	13.28	13.31	13.30	13.29	0.950	0.950	0.937	0.950	0.950	0.936
200	10	200	20	12.93	12.94	12.95	12.94	0.950	0.950	0.944	0.950	0.950	0.943
300	10	300	20	12.84	12.84	12.83	12.83	0.950	0.950	0.945	0.950	0.950	0.945
20	20	20	10	15.71	16.43	16.04	15.66	0.954	0.949	0.890	0.945	0.940	0.874
30	20	30	10	14.69	14.99	14.86	14.74	0.952	0.951	0.909	0.948	0.946	0.903
50	20	50	10	13.90	14.00	13.96	13.93	0.951	0.950	0.925	0.949	0.949	0.923
100	20	100	10	13.29	13.31	13.28	13.28	0.950	0.950	0.937	0.949	0.949	0.936
200	20	200	10	12.95	12.96	12.94	12.94	0.950	0.950	0.943	0.950	0.950	0.943
300	20	300	10	12.82	12.82	12.83	12.83	0.950	0.950	0.946	0.950	0.950	0.946
20	20	20	20	15.49	16.32	16.01	15.46	0.956	0.950	0.894	0.946	0.938	0.876
30	20	30	20	14.62	14.98	14.82	14.63	0.953	0.950	0.911	0.948	0.945	0.903
50	20	50	20	13.86	13.99	13.93	13.89	0.951	0.950	0.926	0.949	0.948	0.923
100	20	100	20	13.27	13.30	13.28	13.27	0.950	0.950	0.937	0.950	0.950	0.936
200	20	200	20	12.95	12.96	12.94	12.94	0.950	0.950	0.944	0.950	0.950	0.943
300	20	300	20	12.82	12.82	12.83	12.82	0.950	0.950	0.946	0.950	0.950	0.946
						$\alpha = 0.01$							
20	10	20	10	22.60	23.46	23.13	22.70	0.991	0.990	0.958	0.989	0.988	0.952
30	10	30	10	20.64	20.95	20.80	20.69	0.990	0.990	0.971	0.990	0.989	0.969
50	10	50	10	19.06	19.17	19.13	19.11	0.990	0.990	0.979	0.990	0.990	0.979
100	10	100	10	17.92	17.94	17.95	17.95	0.990	0.990	0.985	0.990	0.990	0.985
200	10	200	10	17.40	17.40	17.38	17.38	0.990	0.990	0.988	0.990	0.990	0.988
300	10	300	10	17.19	17.19	17.19	17.19	0.990	0.990	0.988	0.990	0.990	0.988
20	10	20	20	22.29	23.33	22.95	22.23	0.991	0.990	0.961	0.989	0.987	0.953
30	10	30	20	20.43	20.84	20.69	20.48	0.991	0.990	0.972	0.990	0.989	0.969
50	10	50	20	19.02	19.15	19.08	19.03	0.990	0.990	0.980	0.990	0.990	0.979
100	10	100	20	17.96	17.99	17.94	17.93	0.990	0.990	0.985	0.990	0.990	0.985
200	10	200	20	17.36	17.37	17.37	17.37	0.990	0.990	0.988	0.990	0.990	0.988
300	10	300	20	17.21	17.21	17.19	17.19	0.990	0.990	0.988	0.990	0.990	0.988
20	20	20	10	22.01	23.09	22.50	21.83	0.991	0.990	0.962	0.989	0.987	0.954
30	20	30	10	20.22	20.66	20.49	20.29	0.991	0.990	0.973	0.990	0.989	0.970
50	20	50	10	18.90	19.05	19.01	18.96	0.990	0.990	0.980	0.990	0.990	0.980
100	20	100	10	17.95	17.98	17.92	17.91	0.990	0.990	0.985	0.990	0.990	0.985
200	20	200	10	17.37	17.38	17.37	17.37	0.990	0.990	0.988	0.990	0.990	0.988
300	20	300	10	17.16	17.16	17.18	17.18	0.990	0.990	0.989	0.990	0.990	0.989
20	20	20	20	21.72	22.96	22.44	21.49	0.992	0.989	0.964	0.989	0.986	0.956
30	20	30	20	20.12	20.64	20.43	20.12	0.991	0.990	0.973	0.989	0.988	0.970
50	20	50	20	18.87	19.06	18.97	18.90	0.990	0.990	0.981	0.990	0.989	0.980
100	20	100	20	17.90	17.95	17.91	17.90	0.990	0.990	0.985	0.990	0.990	0.985
200	20	200	20	17.36	17.37	17.37	17.36	0.990	0.990	0.988	0.990	0.990	0.988
300	20	300	20	17.15	17.16	17.18	17.18	0.990	0.990	0.989	0.990	0.990	0.989

Note. $\text{CP}_{\text{YS}\cdot\text{L}} = \text{CP}(t^2_{\text{YS}\cdot\text{L}}(\alpha))$, $\text{CP}_{\text{YS}\cdot F} = \text{CP}(t^2_{\text{YS}\cdot F}(\alpha))$, $\text{CP}_{\chi^2} = \text{CP}(\chi^2_{p,\alpha})$, $\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha))$, $\widetilde{\text{CP}}_{\text{YS}\cdot F} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha))$, $\widetilde{\text{CP}}_{\chi^2} = \widetilde{\text{CP}}(\chi^2_{p,\alpha})$, $\chi^2_{6,0.05} = 12.59$, $\chi^2_{6,0.01} = 16.81$.

Table 2: Simulated and approximate values and coverage probabilities for $(p_1, p_2, p_1, p_3) = (6, 5, 6, 1)$ in Case I

Sample Size				Upper Percentile				Coverage Probability					
$n_1^{(1)}$	$n_2^{(1)}$	$n_1^{(2)}$	$n_3^{(2)}$	t^2_{simu}	$\widetilde{t}^2_{\text{simu}}$	$t^2_{\text{YS}\cdot\text{L}}$	$t^2_{\text{YS}\cdot F}$	$\text{CP}_{\text{YS}\cdot\text{L}}$	$\text{CP}_{\text{YS}\cdot F}$	CP_{χ^2}	$\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}}$	$\widetilde{\text{CP}}_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\chi^2}$
						$\alpha = 0.05$							
20	10	20	10	16.17	16.55	16.42	16.18	0.953	0.950	0.880	0.948	0.945	0.872
30	10	30	10	14.94	15.08	15.04	14.98	0.952	0.951	0.904	0.950	0.948	0.901
50	10	50	10	13.98	14.03	14.03	14.02	0.951	0.951	0.923	0.950	0.950	0.922
100	10	100	10	13.29	13.30	13.31	13.30	0.950	0.950	0.937	0.950	0.950	0.937
200	10	200	10	12.94	12.94	12.95	12.95	0.950	0.950	0.944	0.950	0.950	0.943
300	10	300	10	12.85	12.85	12.83	12.83	0.950	0.950	0.945	0.950	0.950	0.945
20	10	20	20	16.02	16.48	16.44	16.03	0.955	0.950	0.882	0.950	0.944	0.872
30	10	30	20	14.89	15.07	15.04	14.91	0.952	0.950	0.905	0.950	0.948	0.901
50	10	50	20	13.95	14.02	14.02	14.00	0.951	0.951	0.924	0.950	0.950	0.922
100	10	100	20	13.28	13.30	13.30	13.30	0.950	0.950	0.937	0.950	0.950	0.936
200	10	200	20	12.92	12.92	12.95	12.95	0.950	0.950	0.944	0.950	0.950	0.944
300	10	300	20	12.82	12.82	12.83	12.83	0.950	0.950	0.946	0.950	0.950	0.946
20	20	20	10	15.73	16.21	15.91	15.55	0.952	0.948	0.889	0.946	0.941	0.879
30	20	30	10	14.71	14.90	14.80	14.68	0.951	0.950	0.909	0.948	0.947	0.905
50	20	50	10	13.91	13.97	13.94	13.91	0.950	0.950	0.925	0.949	0.949	0.924
100	20	100	10	13.27	13.29	13.28	13.28	0.950	0.950	0.937	0.950	0.950	0.937
200	20	200	10	12.94	12.95	12.94	12.94	0.950	0.950	0.944	0.950	0.950	0.944
300	20	300	10	12.82	12.82	12.83	12.83	0.950	0.950	0.946	0.950	0.950	0.946
20	20	20	20	15.61	16.17	16.01	15.46	0.955	0.948	0.891	0.948	0.940	0.879
30	20	30	20	14.64	14.87	14.82	14.63	0.953	0.950	0.910	0.949	0.947	0.905
50	20	50	20	13.89	13.97	13.93	13.89	0.951	0.950	0.925	0.949	0.949	0.924
100	20	100	20	13.28	13.30	13.28	13.27	0.950	0.950	0.937	0.950	0.949	0.937
200	20	200	20	12.94	12.94	12.94	12.94	0.950	0.950	0.944	0.950	0.950	0.943
300	20	300	20	12.83	12.83	12.83	12.82	0.950	0.950	0.945	0.950	0.950	0.945
						$\alpha = 0.01$							
20	10	20	10	22.73	23.28	23.13	22.70	0.991	0.990	0.957	0.990	0.989	0.953
30	10	30	10	20.66	20.87	20.80	20.69	0.990	0.990	0.970	0.990	0.990	0.969
50	10	50	10	19.01	19.08	19.13	19.11	0.990	0.990	0.980	0.990	0.990	0.979
100	10	100	10	17.95	17.96	17.95	17.95	0.990	0.990	0.985	0.990	0.990	0.985
200	10	200	10	17.36	17.36	17.38	17.38	0.990	0.990	0.988	0.990	0.990	0.988
300	10	300	10	17.22	17.22	17.19	17.19	0.990	0.990	0.988	0.990	0.990	0.988
20	10	20	20	22.57	23.23	23.18	22.46	0.991	0.990	0.959	0.990	0.988	0.954
30	10	30	20	20.58	20.84	20.79	20.58	0.991	0.990	0.971	0.990	0.989	0.969
50	10	50	20	19.02	19.11	19.12	19.07	0.990	0.990	0.980	0.990	0.990	0.979
100	10	100	20	17.93	17.95	17.95	17.94	0.990	0.990	0.985	0.990	0.990	0.985
200	10	200	20	17.35	17.36	17.38	17.38	0.990	0.990	0.988	0.990	0.990	0.988
300	10	300	20	17.20	17.20	17.19	17.19	0.990	0.990	0.988	0.990	0.990	0.988
20	20	20	10	22.05	22.78	22.27	21.65	0.991	0.989	0.962	0.989	0.987	0.957
30	20	30	10	20.23	20.51	20.39	20.20	0.990	0.990	0.973	0.990	0.989	0.971
50	20	50	10	18.92	19.01	18.97	18.93	0.990	0.990	0.980	0.990	0.990	0.980
100	20	100	10	17.92	17.94	17.91	17.90	0.990	0.990	0.985	0.990	0.990	0.985
200	20	200	10	17.41	17.42	17.37	17.37	0.990	0.990	0.987	0.990	0.990	0.987
300	20	300	10	17.20	17.21	17.18	17.18	0.990	0.990	0.988	0.990	0.990	0.988
20	20	20	20	21.89	22.71	22.44	21.49	0.991	0.989	0.963	0.989	0.986	0.957
30	20	30	20	20.14	20.47	20.43	20.12	0.991	0.990	0.974	0.990	0.989	0.971
50	20	50	20	18.88	18.99	18.97	18.90	0.990	0.990	0.980	0.990	0.990	0.980
100	20	100	20	17.88	17.91	17.91	17.90	0.990	0.990	0.985	0.990	0.990	0.985
200	20	200	20	17.37	17.38	17.37	17.36	0.990	0.990	0.988	0.990	0.990	0.988
300	20	300	20	17.22	17.22	17.18	17.18	0.990	0.990	0.988	0.990	0.990	0.988

Note. $\text{CP}_{\text{YS}\cdot\text{L}} = \text{CP}(t^2_{\text{YS}\cdot\text{L}}(\alpha))$, $\text{CP}_{\text{YS}\cdot F} = \text{CP}(t^2_{\text{YS}\cdot F}(\alpha))$, $\text{CP}_{\chi^2} = \text{CP}(\chi^2_{p,\alpha})$, $\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha))$, $\widetilde{\text{CP}}_{\text{YS}\cdot F} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha))$, $\widetilde{\text{CP}}_{\chi^2} = \widetilde{\text{CP}}(\chi^2_{p,\alpha})$, $\chi^2_{6,0.05} = 12.59$, $\chi^2_{6,0.01} = 16.81$.

Table 3: Simulated and approximate values and coverage probabilities for $(p_1, p_2, p_1, p_3) = (12, 8, 12, 4)$ in Case I

Sample Size				Upper Percentile				Coverage Probability					
$n_1^{(1)}$	$n_2^{(1)}$	$n_1^{(2)}$	$n_3^{(2)}$	t^2_{simu}	$\widetilde{t}^2_{\text{simu}}$	$t^2_{\text{YS}\cdot\text{L}}$	$t^2_{\text{YS}\cdot F}$	$\text{CP}_{\text{YS}\cdot\text{L}}$	$\text{CP}_{\text{YS}\cdot F}$	CP_{χ^2}	$\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}}$	$\widetilde{\text{CP}}_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\chi^2}$
						$\alpha = 0.05$							
20	10	20	10	32.16	35.41	34.00	32.83	0.962	0.955	0.758	0.940	0.930	0.695
30	10	30	10	28.09	29.11	28.66	28.39	0.955	0.953	0.826	0.946	0.943	0.805
50	10	50	10	25.09	25.39	25.27	25.22	0.952	0.951	0.881	0.949	0.948	0.875
100	10	100	10	23.00	23.06	23.05	23.05	0.951	0.951	0.919	0.950	0.950	0.918
200	10	200	10	22.00	22.01	22.02	22.02	0.950	0.950	0.935	0.950	0.950	0.935
300	10	300	10	21.67	21.68	21.68	21.68	0.950	0.950	0.940	0.950	0.950	0.940
20	10	20	20	31.33	35.11	33.68	31.76	0.965	0.953	0.774	0.940	0.922	0.700
30	10	30	20	27.67	28.95	28.44	27.95	0.957	0.953	0.834	0.945	0.940	0.807
50	10	50	20	24.92	25.31	25.17	25.08	0.953	0.952	0.885	0.948	0.947	0.877
100	10	100	20	22.97	23.06	23.02	23.01	0.951	0.951	0.919	0.950	0.949	0.917
200	10	200	20	22.01	22.03	22.01	22.01	0.950	0.950	0.935	0.950	0.950	0.935
300	10	300	20	21.68	21.69	21.68	21.68	0.950	0.950	0.940	0.950	0.950	0.940
20	20	20	10	30.63	34.68	32.62	30.86	0.963	0.952	0.786	0.934	0.917	0.708
30	20	30	10	27.36	28.70	28.03	27.57	0.956	0.952	0.841	0.943	0.938	0.812
50	20	50	10	24.82	25.23	25.03	24.94	0.952	0.951	0.886	0.948	0.947	0.878
100	20	100	10	22.94	23.03	22.99	22.98	0.951	0.951	0.920	0.949	0.949	0.918
200	20	200	10	21.99	22.01	22.00	22.00	0.950	0.950	0.935	0.950	0.950	0.935
300	20	300	10	21.66	21.67	21.68	21.68	0.950	0.950	0.940	0.950	0.950	0.940
20	20	20	20	29.88	34.45	32.56	30.10	0.968	0.952	0.801	0.935	0.911	0.712
30	20	30	20	26.97	28.60	27.91	27.22	0.958	0.952	0.848	0.943	0.936	0.815
50	20	50	20	24.66	25.17	24.96	24.81	0.953	0.952	0.889	0.948	0.946	0.879
100	20	100	20	22.89	23.01	22.96	22.95	0.951	0.951	0.921	0.950	0.949	0.918
200	20	200	20	21.96	21.99	22.00	21.99	0.950	0.950	0.936	0.950	0.950	0.935
300	20	300	20	21.63	21.64	21.67	21.67	0.951	0.951	0.941	0.950	0.950	0.941
						$\alpha = 0.01$							
20	10	20	10	43.43	48.14	45.83	43.90	0.993	0.991	0.882	0.987	0.983	0.838
30	10	30	10	36.72	38.17	37.46	37.04	0.991	0.991	0.930	0.989	0.988	0.917
50	10	50	10	32.17	32.56	32.37	32.30	0.991	0.990	0.961	0.990	0.989	0.958
100	10	100	10	29.07	29.15	29.12	29.12	0.990	0.990	0.978	0.990	0.990	0.978
200	10	200	10	27.59	27.61	27.63	27.63	0.990	0.990	0.985	0.990	0.990	0.985
300	10	300	10	27.18	27.19	27.15	27.15	0.990	0.990	0.987	0.990	0.990	0.987
20	10	20	20	42.24	47.73	45.36	42.22	0.994	0.990	0.892	0.987	0.980	0.841
30	10	30	20	36.16	37.95	37.14	36.38	0.992	0.990	0.934	0.988	0.987	0.919
50	10	50	20	31.89	32.42	32.23	32.08	0.991	0.990	0.963	0.990	0.989	0.959
100	10	100	20	28.99	29.10	29.08	29.07	0.990	0.990	0.979	0.990	0.990	0.978
200	10	200	20	27.58	27.61	27.62	27.62	0.990	0.990	0.985	0.990	0.990	0.985
300	10	300	20	27.11	27.12	27.15	27.15	0.990	0.990	0.987	0.990	0.990	0.987
20	20	20	10	41.09	46.98	43.69	40.82	0.993	0.990	0.901	0.985	0.978	0.848
30	20	30	10	35.63	37.51	36.51	35.80	0.992	0.990	0.938	0.988	0.986	0.922
50	20	50	10	31.73	32.28	32.02	31.88	0.991	0.990	0.963	0.989	0.989	0.960
100	20	100	10	28.95	29.08	29.03	29.02	0.990	0.990	0.979	0.990	0.990	0.978
200	20	200	10	27.56	27.59	27.61	27.61	0.990	0.990	0.985	0.990	0.990	0.985
300	20	300	10	27.15	27.16	27.14	27.14	0.990	0.990	0.987	0.990	0.990	0.987
20	20	20	20	39.99	46.64	43.64	39.65	0.994	0.990	0.911	0.985	0.975	0.851
30	20	30	20	35.21	37.41	36.35	35.28	0.992	0.990	0.942	0.988	0.985	0.923
50	20	50	20	31.48	32.12	31.92	31.70	0.991	0.990	0.965	0.989	0.989	0.960
100	20	100	20	28.93	29.09	29.00	28.97	0.990	0.990	0.979	0.990	0.990	0.978
200	20	200	20	27.60	27.63	27.60	27.60	0.990	0.990	0.985	0.990	0.990	0.985
300	20	300	20	27.04	27.05	27.14	27.14	0.990	0.990	0.987	0.990	0.990	0.987

Note. $\text{CP}_{\text{YS}\cdot\text{L}} = \text{CP}(t^2_{\text{YS}\cdot\text{L}}(\alpha))$, $\text{CP}_{\text{YS}\cdot F} = \text{CP}(t^2_{\text{YS}\cdot F}(\alpha))$, $\text{CP}_{\chi^2} = \text{CP}(\chi^2_{p,\alpha})$, $\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha))$, $\widetilde{\text{CP}}_{\text{YS}\cdot F} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha))$, $\widetilde{\text{CP}}_{\chi^2} = \widetilde{\text{CP}}(\chi^2_{p,\alpha})$, $\chi^2_{12,0.05} = 21.03$, $\chi^2_{12,0.01} = 26.22$.

Table 4: Simulated and approximate values and coverage probabilities for $(p_1, p_2, p_1, p_3) = (12, 10, 12, 2)$ in Case I

Sample Size				Upper Percentile				Coverage Probability					
$n_1^{(1)}$	$n_2^{(1)}$	$n_1^{(2)}$	$n_3^{(2)}$	t^2_{simu}	$\widetilde{t}^2_{\text{simu}}$	$t^2_{\text{YS}\cdot\text{L}}$	$t^2_{\text{YS}\cdot F}$	$\text{CP}_{\text{YS}\cdot\text{L}}$	$\text{CP}_{\text{YS}\cdot F}$	CP_{χ^2}	$\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}}$	$\widetilde{\text{CP}}_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\chi^2}$
						$\alpha = 0.05$							
20	10	20	10	32.35	34.66	34.00	32.83	0.961	0.953	0.752	0.946	0.936	0.708
30	10	30	10	28.21	28.90	28.66	28.39	0.954	0.952	0.823	0.948	0.945	0.809
50	10	50	10	25.13	25.32	25.27	25.22	0.952	0.951	0.880	0.949	0.949	0.876
100	10	100	10	23.04	23.08	23.05	23.05	0.950	0.950	0.918	0.950	0.950	0.917
200	10	200	10	22.01	22.02	22.02	22.02	0.950	0.950	0.935	0.950	0.950	0.935
300	10	300	10	21.68	21.68	21.68	21.68	0.950	0.950	0.940	0.950	0.950	0.940
20	10	20	20	31.92	34.54	34.21	32.27	0.964	0.953	0.761	0.948	0.932	0.710
30	10	30	20	28.02	28.86	28.65	28.16	0.956	0.951	0.828	0.948	0.943	0.811
50	10	50	20	25.07	25.32	25.24	25.15	0.952	0.951	0.881	0.949	0.948	0.876
100	10	100	20	23.01	23.07	23.04	23.03	0.950	0.950	0.918	0.950	0.950	0.917
200	10	200	20	22.02	22.03	22.01	22.01	0.950	0.950	0.935	0.950	0.950	0.935
300	10	300	20	21.68	21.68	21.68	21.68	0.950	0.950	0.940	0.950	0.950	0.940
20	20	20	10	30.57	33.49	32.09	30.46	0.961	0.949	0.784	0.939	0.923	0.728
30	20	30	10	27.41	28.33	27.82	27.39	0.954	0.950	0.839	0.945	0.941	0.820
50	20	50	10	24.84	25.11	24.96	24.88	0.951	0.950	0.886	0.948	0.947	0.881
100	20	100	10	22.93	22.99	22.97	22.96	0.950	0.950	0.919	0.950	0.950	0.918
200	20	200	10	21.97	21.99	22.00	22.00	0.950	0.950	0.936	0.950	0.950	0.935
300	20	300	10	21.68	21.69	21.67	21.67	0.950	0.950	0.940	0.950	0.950	0.940
20	20	20	20	30.16	33.39	32.56	30.10	0.966	0.950	0.793	0.944	0.920	0.730
30	20	30	20	27.19	28.27	27.91	27.22	0.957	0.950	0.843	0.947	0.939	0.821
50	20	50	20	24.74	25.07	24.96	24.81	0.952	0.951	0.888	0.949	0.947	0.881
100	20	100	20	22.95	23.02	22.96	22.95	0.950	0.950	0.919	0.949	0.949	0.918
200	20	200	20	21.97	21.99	22.00	21.99	0.950	0.950	0.935	0.950	0.950	0.935
300	20	300	20	21.67	21.68	21.67	21.67	0.950	0.950	0.940	0.950	0.950	0.940
						$\alpha = 0.01$							
20	10	20	10	43.49	46.89	45.83	43.90	0.993	0.991	0.879	0.988	0.985	0.848
30	10	30	10	36.83	37.78	37.46	37.04	0.991	0.990	0.928	0.989	0.989	0.920
50	10	50	10	32.19	32.44	32.37	32.30	0.990	0.990	0.961	0.990	0.990	0.959
100	10	100	10	29.10	29.15	29.12	29.12	0.990	0.990	0.978	0.990	0.990	0.978
200	10	200	10	27.67	27.68	27.63	27.63	0.990	0.990	0.985	0.990	0.990	0.985
300	10	300	10	27.18	27.18	27.15	27.15	0.990	0.990	0.987	0.990	0.990	0.987
20	10	20	20	42.90	46.72	46.19	43.02	0.994	0.990	0.885	0.989	0.984	0.849
30	10	30	20	36.57	37.71	37.46	36.70	0.992	0.990	0.931	0.989	0.988	0.921
50	10	50	20	32.07	32.41	32.33	32.19	0.991	0.990	0.961	0.990	0.990	0.959
100	10	100	20	29.07	29.13	29.11	29.09	0.990	0.990	0.978	0.990	0.990	0.978
200	10	200	20	27.60	27.62	27.63	27.63	0.990	0.990	0.985	0.990	0.990	0.985
300	10	300	20	27.12	27.12	27.15	27.15	0.990	0.990	0.987	0.990	0.990	0.987
20	20	20	10	40.78	45.11	42.86	40.21	0.993	0.989	0.901	0.986	0.980	0.863
30	20	30	10	35.61	36.91	36.20	35.53	0.991	0.990	0.937	0.989	0.987	0.927
50	20	50	10	31.71	32.06	31.92	31.79	0.991	0.990	0.963	0.990	0.989	0.961
100	20	100	10	28.94	29.02	29.01	28.99	0.990	0.990	0.979	0.990	0.990	0.979
200	20	200	10	27.58	27.60	27.61	27.60	0.990	0.990	0.985	0.990	0.990	0.985
300	20	300	10	27.13	27.14	27.14	27.14	0.990	0.990	0.987	0.990	0.990	0.987
20	20	20	20	40.16	44.97	43.64	39.65	0.994	0.989	0.906	0.988	0.979	0.864
30	20	30	20	35.33	36.83	36.35	35.28	0.992	0.990	0.940	0.989	0.987	0.927
50	20	50	20	31.61	32.04	31.92	31.70	0.991	0.990	0.964	0.990	0.989	0.961
100	20	100	20	29.01	29.11	29.00	28.97	0.990	0.990	0.979	0.990	0.990	0.978
200	20	200	20	27.63	27.65	27.60	27.60	0.990	0.990	0.985	0.990	0.990	0.985
300	20	300	20	27.13	27.14	27.14	27.14	0.990	0.990	0.987	0.990	0.990	0.987

Note. $\text{CP}_{\text{YS}\cdot\text{L}} = \text{CP}(t^2_{\text{YS}\cdot\text{L}}(\alpha))$, $\text{CP}_{\text{YS}\cdot F} = \text{CP}(t^2_{\text{YS}\cdot F}(\alpha))$, $\text{CP}_{\chi^2} = \text{CP}(\chi^2_{p,\alpha})$, $\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha))$, $\widetilde{\text{CP}}_{\text{YS}\cdot F} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha))$, $\widetilde{\text{CP}}_{\chi^2} = \widetilde{\text{CP}}(\chi^2_{p,\alpha})$, $\chi^2_{12,0.05} = 21.03$, $\chi^2_{12,0.01} = 26.22$.

Table 5: Simulated and approximate values and coverage probabilities for $(p_1, p_2) = (6, 3)$ in Case II

Sample Size				Upper Percentile				Coverage Probability					
$n_1^{(1)}$	$n_2^{(1)}$	$n_1^{(2)}$	$n_3^{(2)}$	$t^2_{\rm simu}$	$\widetilde{t}^2_{\rm simu}$	$t^2_{\rm YS\cdot L}$	$t^2_{{\rm YS}\cdot F}$	${\rm CP}_{\rm YS\cdot L}$	${\rm CP}_{{\rm YS}\cdot F}$	${\rm CP}_{\chi^2}$	$\widetilde{\rm CP}_{\rm YS\cdot L}$	$\widetilde{\rm CP}_{{\rm YS}\cdot F}$	$\widetilde{\rm CP}_{\chi^2}$
$\alpha = 0.05$													
20	10	20	10	16.07	16.68	16.42	16.18	0.954	0.951	0.882	0.947	0.944	0.869
30	10	30	10	14.91	15.15	15.04	14.98	0.952	0.951	0.905	0.949	0.948	0.899
50	10	50	10	13.99	14.07	14.03	14.02	0.951	0.950	0.923	0.949	0.949	0.921
100	10	100	10	13.28	13.30	13.31	13.30	0.950	0.950	0.937	0.950	0.950	0.937
200	10	200	10	12.97	12.97	12.95	12.95	0.950	0.950	0.943	0.950	0.950	0.943
300	10	300	10	12.84	12.84	12.83	12.83	0.950	0.950	0.945	0.950	0.950	0.945
20	10	20	20	15.81	16.57	16.18	15.78	0.954	0.950	0.887	0.945	0.939	0.871
30	10	30	20	14.75	15.06	14.92	14.79	0.952	0.951	0.908	0.948	0.946	0.901
50	10	50	20	13.93	14.04	13.98	13.95	0.951	0.950	0.924	0.949	0.949	0.922
100	10	100	20	13.30	13.33	13.29	13.29	0.950	0.950	0.937	0.949	0.949	0.936
200	10	200	20	12.92	12.93	12.94	12.94	0.950	0.950	0.944	0.950	0.950	0.944
300	10	300	20	12.84	12.84	12.83	12.83	0.950	0.950	0.945	0.950	0.950	0.945
20	20	20	20	15.48	16.37	16.01	15.46	0.956	0.950	0.895	0.945	0.938	0.875
30	20	30	20	14.58	14.97	14.82	14.63	0.953	0.951	0.911	0.948	0.945	0.903
50	20	50	20	13.87	14.01	13.93	13.89	0.951	0.950	0.926	0.949	0.948	0.923
100	20	100	20	13.25	13.28	13.28	13.27	0.951	0.950	0.938	0.950	0.950	0.937
200	20	200	20	12.92	12.93	12.94	12.94	0.950	0.950	0.944	0.950	0.950	0.944
300	20	300	20	12.82	12.82	12.83	12.82	0.950	0.950	0.946	0.950	0.950	0.946
$\alpha = 0.01$													
20	10	20	10	22.65	23.56	23.13	22.70	0.991	0.990	0.958	0.989	0.988	0.951
30	10	30	10	20.58	20.92	20.80	20.69	0.991	0.990	0.971	0.990	0.989	0.968
50	10	50	10	19.09	19.20	19.13	19.11	0.990	0.990	0.979	0.990	0.990	0.979
100	10	100	10	17.90	17.93	17.95	17.95	0.990	0.990	0.985	0.990	0.990	0.985
200	10	200	10	17.40	17.41	17.38	17.38	0.990	0.990	0.988	0.990	0.990	0.988
300	10	300	10	17.22	17.23	17.19	17.19	0.990	0.990	0.988	0.990	0.990	0.988
20	10	20	20	22.20	23.33	22.73	22.02	0.991	0.990	0.961	0.988	0.986	0.953
30	10	30	20	20.37	20.82	20.59	20.38	0.991	0.990	0.972	0.989	0.989	0.969
50	10	50	20	18.97	19.12	19.04	19.00	0.990	0.990	0.980	0.990	0.990	0.979
100	10	100	20	17.90	17.94	17.93	17.92	0.990	0.990	0.985	0.990	0.990	0.985
200	10	200	20	17.33	17.34	17.37	17.37	0.990	0.990	0.988	0.990	0.990	0.988
300	10	300	20	17.15	17.15	17.19	17.19	0.990	0.990	0.989	0.990	0.990	0.989
20	20	20	20	21.72	23.06	22.44	21.49	0.992	0.989	0.965	0.988	0.985	0.955
30	20	30	20	20.13	20.68	20.43	20.12	0.991	0.990	0.974	0.989	0.988	0.970
50	20	50	20	18.88	19.07	18.97	18.90	0.990	0.990	0.980	0.990	0.989	0.979
100	20	100	20	17.88	17.93	17.91	17.90	0.990	0.990	0.985	0.990	0.990	0.985
200	20	200	20	17.33	17.34	17.37	17.36	0.990	0.990	0.988	0.990	0.990	0.988
300	20	300	20	17.17	17.17	17.18	17.18	0.990	0.990	0.989	0.990	0.990	0.988

Note. ${\rm CP}_{\rm YS\cdot L} = {\rm CP}(t^2_{\rm YS\cdot L}(\alpha))$, ${\rm CP}_{{\rm YS}\cdot F} = {\rm CP}(t^2_{{\rm YS}\cdot F}(\alpha))$, ${\rm CP}_{\chi^2} = {\rm CP}(\chi^2_{p,\alpha})$, $\widetilde{\rm CP}_{\rm YS\cdot L} = \widetilde{\rm CP}(t^2_{\rm YS\cdot L}(\alpha))$, $\widetilde{\rm CP}_{{\rm YS}\cdot F} = \widetilde{\rm CP}(t^2_{{\rm YS}\cdot F}(\alpha))$, $\widetilde{\rm CP}_{\chi^2} = \widetilde{\rm CP}(\chi^2_{p,\alpha})$, $\chi^2_{6,0.05} = 12.59$, $\chi^2_{6,0.01} = 16.81$.

Table 6: Simulated and approximate values and coverage probabilities for $(p_1, p_2) = (12, 6)$ in Case II

Sample Size				Upper Percentile				Coverage Probability					
$n_1^{(1)}$	$n_2^{(1)}$	$n_1^{(2)}$	$n_3^{(2)}$	t^2_{simu}	$\widetilde{t}^2_{\text{simu}}$	$t^2_{\text{YS}\cdot\text{L}}$	$t^2_{\text{YS}\cdot F}$	$\text{CP}_{\text{YS}\cdot\text{L}}$	$\text{CP}_{\text{YS}\cdot F}$	CP_{χ^2}	$\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}}$	$\widetilde{\text{CP}}_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\chi^2}$
$\alpha = 0.05$													
20	10	20	10	32.20	35.59	34.00	32.83	0.961	0.954	0.759	0.939	0.929	0.694
30	10	30	10	28.07	29.20	28.66	28.39	0.955	0.953	0.828	0.945	0.942	0.804
50	10	50	10	25.07	25.40	25.27	25.22	0.952	0.952	0.882	0.949	0.948	0.875
100	10	100	10	23.03	23.10	23.05	23.05	0.950	0.950	0.918	0.949	0.949	0.917
200	10	200	10	22.05	22.06	22.02	22.02	0.950	0.950	0.934	0.949	0.949	0.934
300	10	300	10	21.70	21.71	21.68	21.68	0.950	0.950	0.940	0.950	0.950	0.940
20	10	20	20	31.04	35.13	33.15	31.29	0.964	0.952	0.780	0.935	0.918	0.701
30	10	30	20	27.54	28.97	28.23	27.75	0.956	0.952	0.837	0.943	0.938	0.807
50	10	50	20	24.89	25.32	25.10	25.01	0.952	0.951	0.885	0.947	0.946	0.876
100	10	100	20	22.96	23.06	23.01	22.99	0.951	0.950	0.919	0.949	0.949	0.917
200	10	200	20	21.97	21.99	22.01	22.00	0.951	0.951	0.935	0.950	0.950	0.935
300	10	300	20	21.70	21.71	21.68	21.68	0.950	0.950	0.940	0.950	0.950	0.940
20	20	20	20	29.92	34.63	32.56	30.10	0.967	0.951	0.800	0.934	0.909	0.709
30	20	30	20	26.94	28.65	27.91	27.22	0.959	0.953	0.848	0.943	0.935	0.812
50	20	50	20	24.66	25.21	24.96	24.81	0.953	0.952	0.889	0.947	0.945	0.878
100	20	100	20	22.88	23.01	22.96	22.95	0.951	0.951	0.921	0.949	0.949	0.918
200	20	200	20	21.99	22.02	22.00	21.99	0.950	0.950	0.935	0.950	0.950	0.935
300	20	300	20	21.68	21.69	21.67	21.67	0.950	0.950	0.940	0.950	0.950	0.940
$\alpha = 0.01$													
20	10	20	10	43.48	48.43	45.83	43.90	0.993	0.991	0.883	0.986	0.983	0.836
30	10	30	10	36.73	38.27	37.46	37.04	0.991	0.991	0.930	0.988	0.988	0.916
50	10	50	10	32.11	32.55	32.37	32.30	0.991	0.990	0.961	0.990	0.989	0.958
100	10	100	10	29.08	29.18	29.12	29.12	0.990	0.990	0.978	0.990	0.990	0.978
200	10	200	10	27.69	27.72	27.63	27.63	0.990	0.990	0.985	0.990	0.990	0.984
300	10	300	10	27.15	27.16	27.15	27.15	0.990	0.990	0.987	0.990	0.990	0.987
20	10	20	20	41.79	47.77	44.53	41.49	0.993	0.990	0.896	0.985	0.978	0.842
30	10	30	20	35.89	37.86	36.83	36.08	0.992	0.990	0.936	0.988	0.986	0.919
50	10	50	20	31.82	32.40	32.13	31.98	0.991	0.990	0.963	0.989	0.989	0.959
100	10	100	20	28.98	29.10	29.06	29.04	0.990	0.990	0.979	0.990	0.990	0.978
200	10	200	20	27.59	27.63	27.62	27.61	0.990	0.990	0.985	0.990	0.990	0.985
300	10	300	20	27.16	27.17	27.15	27.15	0.990	0.990	0.987	0.990	0.990	0.986
20	20	20	20	40.10	46.96	43.64	39.65	0.994	0.989	0.910	0.985	0.974	0.848
30	20	30	20	35.09	37.44	36.35	35.28	0.992	0.990	0.943	0.988	0.985	0.923
50	20	50	20	31.58	32.31	31.92	31.70	0.991	0.990	0.965	0.989	0.988	0.960
100	20	100	20	28.89	29.06	29.00	28.97	0.990	0.990	0.979	0.990	0.990	0.978
200	20	200	20	27.56	27.60	27.60	27.60	0.990	0.990	0.985	0.990	0.990	0.985
300	20	300	20	27.11	27.13	27.14	27.14	0.990	0.990	0.987	0.990	0.990	0.987

Note. $\text{CP}_{\text{YS}\cdot\text{L}} = \text{CP}(t^2_{\text{YS}\cdot\text{L}}(\alpha))$, $\text{CP}_{\text{YS}\cdot F} = \text{CP}(t^2_{\text{YS}\cdot F}(\alpha))$, $\text{CP}_{\chi^2} = \text{CP}(\chi^2_{p,\alpha})$, $\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha))$, $\widetilde{\text{CP}}_{\text{YS}\cdot F} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha))$, $\widetilde{\text{CP}}_{\chi^2} = \widetilde{\text{CP}}(\chi^2_{p,\alpha})$, $\chi^2_{12,0.05} = 21.03$, $\chi^2_{12,0.01} = 26.22$.

Table 7: Simulated and approximate values and coverage probabilities for comparisons with a control ($m = 3$ and $(p_1, p_2, p_3) = (6, 4, 2)$)

Sample Size			Upper Percentile				Coverage Probability		
$n_1^{(\ell)}$	$n_2^{(\ell)}$	$n_3^{(\ell)}$	$\widetilde{t}^2_{\text{simu}\cdot\text{c}}$	$\widetilde{t}^2_{\text{simu}\cdot\text{Bon}}$	$t^2_{\text{YS}\cdot\text{L}}$	$t^2_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}}$	$\widetilde{\text{CP}}_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\chi^2}$
				$\alpha = 0.05$					
13	5	5	19.35	19.62	19.66	19.00	0.953	0.946	0.848
15	5	5	18.63	18.89	18.86	18.43	0.953	0.948	0.863
20	5	5	17.50	17.74	17.67	17.49	0.952	0.950	0.888
25	5	5	16.82	17.03	17.00	16.90	0.952	0.951	0.902
13	10	10	18.60	18.90	19.08	17.81	0.956	0.939	0.865
15	10	10	18.01	18.28	18.37	17.49	0.955	0.943	0.877
20	10	10	17.10	17.32	17.33	16.90	0.953	0.947	0.896
25	10	10	16.56	16.75	16.75	16.51	0.953	0.949	0.908
30	10	10	16.19	16.38	16.37	16.22	0.953	0.951	0.916
50	10	10	15.46	15.62	15.63	15.59	0.953	0.952	0.930
100	10	10	14.89	15.05	15.06	15.05	0.953	0.953	0.942
200	10	10	14.59	14.77	14.76	14.76	0.953	0.953	0.947
30	20	20	15.93	16.15	16.16	15.84	0.953	0.949	0.921
50	20	20	15.33	15.53	15.51	15.42	0.953	0.951	0.933
100	20	20	14.85	15.01	15.02	15.00	0.953	0.953	0.942
200	20	20	14.60	14.74	14.75	14.74	0.953	0.953	0.947
				$\alpha = 0.01$					
13	5	5	26.55	26.67	26.61	25.53	0.990	0.988	0.940
15	5	5	25.32	25.44	25.33	24.63	0.990	0.988	0.949
20	5	5	23.41	23.56	23.44	23.15	0.990	0.989	0.962
25	5	5	22.33	22.40	22.40	22.25	0.990	0.990	0.970
13	10	10	25.34	25.57	25.72	23.65	0.991	0.985	0.949
15	10	10	24.30	24.47	24.57	23.15	0.991	0.987	0.956
20	10	10	22.79	22.86	22.92	22.25	0.990	0.988	0.967
25	10	10	21.94	21.98	22.02	21.65	0.990	0.989	0.972
30	10	10	21.38	21.45	21.44	21.21	0.990	0.989	0.976
50	10	10	20.25	20.32	20.31	20.25	0.990	0.990	0.982
100	10	10	19.36	19.42	19.45	19.44	0.990	0.990	0.986
200	10	10	18.97	19.06	19.01	19.01	0.990	0.990	0.988
30	20	20	20.96	21.09	21.12	20.62	0.990	0.989	0.978
50	20	20	20.07	20.16	20.14	19.99	0.990	0.990	0.983
100	20	20	19.31	19.37	19.39	19.37	0.990	0.990	0.987
200	20	20	18.98	19.05	18.99	18.99	0.990	0.990	0.988

Note. $\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha_\text{c}))$, $\widetilde{\text{CP}}_{\text{YS}\cdot F} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha_\text{c}))$, $\widetilde{\text{CP}}_{\chi^2} = \widetilde{\text{CP}}(\chi^2_{p,\alpha_\text{c}})$, $\chi^2_{6,\alpha_\text{c}} = 14.45$ $(\alpha = 0.05)$, $\chi^2_{6,\alpha_\text{c}} = 18.55$ $(\alpha = 0.01)$, $\alpha_\text{c} = \alpha/(m-1)$.

Table 8: Simulated and approximate values and coverage probabilities for comparisons with a control ($m = 3$ and $(p_1, p_2, p_3) = (12, 8, 4)$)

Sample Size			Upper Percentile				Coverage Probability		
$n_1^{(\ell)}$	$n_2^{(\ell)}$	$n_3^{(\ell)}$	$\widetilde{t}^2_{\mathrm{simu}\cdot\mathrm{c}}$	$\widetilde{t}^2_{\mathrm{simu}\cdot\mathrm{Bon}}$	$t^2_{\mathrm{YS}\cdot\mathrm{L}}$	$t^2_{\mathrm{YS}\cdot F}$	$\widetilde{\mathrm{CP}}_{\mathrm{YS}\cdot\mathrm{L}}$	$\widetilde{\mathrm{CP}}_{\mathrm{YS}\cdot F}$	$\widetilde{\mathrm{CP}}_{\chi^2}$
				$\alpha = 0.05$					
13	5	5	38.39	38.97	40.03	37.07	0.959	0.941	0.668
15	5	5	36.13	36.66	36.84	35.08	0.955	0.942	0.704
20	5	5	32.40	32.73	32.61	31.95	0.952	0.946	0.773
25	5	5	30.33	30.64	30.45	30.13	0.951	0.948	0.816
13	10	10	35.18	35.68	38.22	32.98	0.968	0.931	0.732
15	10	10	33.81	34.21	35.34	31.95	0.961	0.933	0.752
20	10	10	31.18	31.56	31.60	30.13	0.954	0.939	0.799
25	10	10	29.49	29.80	29.72	28.95	0.952	0.944	0.834
30	10	10	28.41	28.70	28.58	28.12	0.952	0.947	0.856
50	10	10	26.28	26.50	26.44	26.34	0.952	0.951	0.899
100	10	10	24.76	24.95	24.90	24.89	0.952	0.952	0.927
200	10	10	23.96	24.12	24.13	24.13	0.952	0.952	0.940
30	20	20	27.74	27.96	28.00	27.02	0.953	0.941	0.870
50	20	20	26.03	26.24	26.14	25.86	0.951	0.948	0.904
100	20	20	24.60	24.81	24.80	24.75	0.953	0.952	0.930
200	20	20	23.92	24.14	24.10	24.09	0.952	0.952	0.941
				$\alpha = 0.01$					
13	5	5	51.23	51.61	52.47	47.84	0.991	0.985	0.817
15	5	5	47.37	47.73	47.60	44.90	0.990	0.986	0.848
20	5	5	41.47	41.62	41.33	40.35	0.990	0.988	0.899
25	5	5	38.31	38.50	38.22	37.76	0.990	0.989	0.926
13	10	10	46.62	46.87	49.88	41.84	0.994	0.981	0.865
15	10	10	44.00	44.09	45.46	40.35	0.992	0.982	0.881
20	10	10	39.79	39.99	39.90	37.76	0.990	0.985	0.915
25	10	10	37.20	37.32	37.20	36.08	0.990	0.987	0.936
30	10	10	35.47	35.60	35.56	34.90	0.990	0.989	0.949
50	10	10	32.51	32.58	32.57	32.42	0.990	0.990	0.970
100	10	10	30.41	30.45	30.44	30.42	0.990	0.990	0.981
200	10	10	29.28	29.36	29.38	29.38	0.990	0.990	0.986
30	20	20	34.60	34.68	34.76	33.37	0.990	0.987	0.956
50	20	20	32.15	32.26	32.15	31.76	0.990	0.989	0.972
100	20	20	30.18	30.25	30.29	30.23	0.990	0.990	0.983
200	20	20	29.29	29.42	29.33	29.32	0.990	0.990	0.986

Note. $\widetilde{\mathrm{CP}}_{\mathrm{YS}\cdot\mathrm{L}} = \widetilde{\mathrm{CP}}(t^2_{\mathrm{YS}\cdot\mathrm{L}}(\alpha_{\mathrm{c}}))$, $\widetilde{\mathrm{CP}}_{\mathrm{YS}\cdot F} = \widetilde{\mathrm{CP}}(t^2_{\mathrm{YS}\cdot F}(\alpha_{\mathrm{c}}))$, $\widetilde{\mathrm{CP}}_{\chi^2} = \widetilde{\mathrm{CP}}(\chi^2_{p,\alpha_{\mathrm{c}}})$, $\chi^2_{12,\alpha_{\mathrm{c}}} = 23.34$ ($\alpha = 0.05$), $\chi^2_{12,\alpha_{\mathrm{c}}} = 28.30$ ($\alpha = 0.01$), $\alpha_{\mathrm{c}} = \alpha/(m-1)$.

Table 9: Simulated and approximate values and coverage probabilities for comparisons with a control ($m = 6$ and $(p_1, p_2, p_3) = (6, 4, 2)$)

Sample Size			Upper Percentile				Coverage Probability		
$n_1^{(\ell)}$	$n_2^{(\ell)}$	$n_3^{(\ell)}$	$\widetilde{t}^2_{\text{simu}\cdot\text{c}}$	$\widetilde{t}^2_{\text{simu}\cdot\text{Bon}}$	$t^2_{\text{YS}\cdot\text{L}}$	$t^2_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}}$	$\widetilde{\text{CP}}_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\chi^2}$
				$\alpha = 0.05$					
13	5	5	19.53	20.06	20.38	20.00	0.961	0.956	0.894
15	5	5	19.19	19.62	19.89	19.63	0.959	0.956	0.902
20	5	5	18.57	19.06	19.11	19.00	0.957	0.956	0.914
25	5	5	18.21	18.62	18.66	18.60	0.957	0.956	0.923
13	10	10	18.88	19.38	19.98	19.21	0.964	0.955	0.909
15	10	10	18.69	19.13	19.55	19.00	0.962	0.955	0.913
20	10	10	18.31	18.74	18.88	18.60	0.958	0.954	0.920
25	10	10	18.00	18.41	18.48	18.32	0.957	0.955	0.927
30	10	10	17.78	18.20	18.22	18.12	0.957	0.955	0.932
50	10	10	17.31	17.68	17.69	17.66	0.956	0.956	0.941
100	10	10	16.90	17.27	17.27	17.27	0.956	0.956	0.948
200	10	10	16.70	17.06	17.05	17.05	0.956	0.956	0.952
30	20	20	17.60	18.00	18.07	17.84	0.957	0.954	0.935
50	20	20	17.22	17.60	17.61	17.54	0.956	0.955	0.943
100	20	20	16.89	17.25	17.24	17.23	0.956	0.956	0.949
200	20	20	16.68	17.01	17.04	17.04	0.956	0.956	0.952
				$\alpha = 0.01$					
13	5	5	25.10	25.40	25.80	25.25	0.992	0.990	0.965
15	5	5	24.56	24.90	25.10	24.73	0.991	0.990	0.969
20	5	5	23.68	23.88	24.00	23.84	0.991	0.991	0.975
25	5	5	23.09	23.35	23.36	23.28	0.991	0.991	0.978
13	10	10	24.17	24.50	25.24	24.14	0.993	0.990	0.972
15	10	10	23.86	24.15	24.62	23.84	0.992	0.990	0.974
20	10	10	23.28	23.52	23.67	23.28	0.991	0.990	0.977
25	10	10	22.84	23.01	23.11	22.89	0.991	0.990	0.980
30	10	10	22.47	22.61	22.75	22.61	0.991	0.990	0.982
50	10	10	21.83	22.06	22.01	21.97	0.991	0.990	0.985
100	10	10	21.26	21.37	21.42	21.42	0.991	0.991	0.988
200	10	10	20.95	21.10	21.12	21.12	0.991	0.991	0.989
30	20	20	22.21	22.40	22.53	22.22	0.991	0.990	0.983
50	20	20	21.68	21.76	21.89	21.79	0.991	0.990	0.986
100	20	20	21.22	21.46	21.38	21.37	0.991	0.991	0.988
200	20	20	20.93	21.04	21.10	21.10	0.991	0.991	0.989

Note. $\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha_\text{c}))$, $\widetilde{\text{CP}}_{\text{YS}\cdot F} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha_\text{c}))$, $\widetilde{\text{CP}}_{\chi^2} = \widetilde{\text{CP}}(\chi^2_{p,\alpha_\text{c}})$, $\chi^2_{6,\alpha_\text{c}} = 16.81$ $(\alpha = 0.05)$, $\chi^2_{6,\alpha_\text{c}} = 20.79$ $(\alpha = 0.01)$, $\alpha_\text{c} = \alpha/(m-1)$.

Table 10: Simulated and approximate values and coverage probabilities for comparisons with a control ($m = 6$ and $(p_1, p_2, p_3) = (12, 8, 4)$)

Sample Size			Upper Percentile				Coverage Probability		
$n_1^{(\ell)}$	$n_2^{(\ell)}$	$n_3^{(\ell)}$	$\widetilde{t}^2_{\text{simu}\cdot\text{c}}$	$\widetilde{t}^2_{\text{simu}\cdot\text{Bon}}$	$t^2_{\text{YS}\cdot\text{L}}$	$t^2_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}}$	$\widetilde{\text{CP}}_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\chi^2}$
				$\alpha = 0.05$					
13	5	5	31.45	32.09	35.09	33.99	0.977	0.971	0.854
15	5	5	31.41	32.10	33.74	33.02	0.970	0.965	0.852
20	5	5	30.66	31.22	31.72	31.41	0.961	0.958	0.866
25	5	5	29.85	30.37	30.58	30.42	0.958	0.956	0.883
13	10	10	29.29	29.91	34.10	31.96	0.984	0.973	0.899
15	10	10	29.74	30.28	32.91	31.41	0.977	0.966	0.888
20	10	10	29.67	30.16	31.14	30.42	0.965	0.958	0.888
25	10	10	29.26	29.69	30.15	29.74	0.960	0.956	0.895
30	10	10	28.81	29.32	29.51	29.26	0.958	0.955	0.904
50	10	10	27.74	28.14	28.24	28.17	0.956	0.956	0.925
100	10	10	26.86	27.31	27.26	27.25	0.955	0.955	0.940
200	10	10	26.36	26.71	26.75	26.75	0.956	0.955	0.948
30	20	20	28.32	28.78	29.15	28.60	0.960	0.954	0.914
50	20	20	27.57	27.94	28.04	27.88	0.956	0.954	0.928
100	20	20	26.79	27.22	27.19	27.16	0.955	0.955	0.941
200	20	20	26.33	26.66	26.73	26.72	0.956	0.956	0.948
				$\alpha = 0.01$					
13	5	5	38.79	39.15	42.71	41.20	0.996	0.994	0.945
15	5	5	38.58	38.95	40.89	39.91	0.994	0.993	0.945
20	5	5	37.35	37.52	38.18	37.76	0.992	0.991	0.953
25	5	5	36.17	36.40	36.66	36.45	0.991	0.991	0.962
13	10	10	35.97	36.28	41.41	38.48	0.997	0.995	0.966
15	10	10	36.31	36.59	39.78	37.76	0.996	0.993	0.963
20	10	10	36.00	36.24	37.41	36.45	0.993	0.991	0.964
25	10	10	35.36	35.49	36.10	35.56	0.992	0.991	0.968
30	10	10	34.73	34.97	35.25	34.92	0.991	0.991	0.972
50	10	10	33.30	33.45	33.58	33.50	0.991	0.991	0.980
100	10	10	32.13	32.31	32.31	32.30	0.991	0.991	0.985
200	10	10	31.49	31.60	31.65	31.65	0.991	0.991	0.988
30	20	20	34.11	34.21	34.78	34.05	0.992	0.990	0.976
50	20	20	33.04	33.17	33.33	33.11	0.991	0.990	0.981
100	20	20	32.05	32.19	32.22	32.18	0.991	0.990	0.986
200	20	20	31.41	31.61	31.62	31.61	0.991	0.991	0.988

Note. $\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha_\text{c}))$, $\widetilde{\text{CP}}_{\text{YS}\cdot F} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha_\text{c}))$, $\widetilde{\text{CP}}_{\chi^2} = \widetilde{\text{CP}}(\chi^2_{p,\alpha_\text{c}})$, $\chi^2_{12,\alpha_\text{c}} = 26.22$ $(\alpha = 0.05)$, $\chi^2_{12,\alpha_\text{c}} = 30.96$ $(\alpha = 0.01)$, $\alpha_\text{c} = \alpha/(m-1)$.

Table 11: Simulated and approximate values and coverage probabilities for comparisons with a control ($m = 10$ and $(p_1, p_2, p_3) = (6, 4, 2)$)

Sample Size			Upper Percentile				Coverage Probability		
$n_1^{(\ell)}$	$n_2^{(\ell)}$	$n_3^{(\ell)}$	$\widetilde{t}^2_{\text{simu}\cdot\text{c}}$	$\widetilde{t}^2_{\text{simu}\cdot\text{Bon}}$	$t^2_{\text{YS}\cdot\text{L}}$	$t^2_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}}$	$\widetilde{\text{CP}}_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\chi^2}$
				$\alpha = 0.05$					
13	5	5	19.98	20.54	21.11	20.83	0.965	0.962	0.916
15	5	5	19.81	20.47	20.74	20.55	0.963	0.960	0.920
20	5	5	19.48	20.00	20.14	20.05	0.960	0.958	0.927
25	5	5	19.21	19.76	19.79	19.74	0.959	0.958	0.932
13	10	10	19.42	19.95	20.80	20.22	0.968	0.961	0.928
15	10	10	19.39	19.88	20.47	20.05	0.965	0.960	0.929
20	10	10	19.16	19.71	19.95	19.74	0.962	0.959	0.933
25	10	10	18.98	19.53	19.64	19.52	0.960	0.958	0.937
30	10	10	18.85	19.38	19.43	19.35	0.959	0.958	0.940
50	10	10	18.51	19.04	19.01	18.99	0.958	0.958	0.946
100	10	10	18.18	18.66	18.66	18.66	0.958	0.958	0.952
200	10	10	18.00	18.49	18.48	18.48	0.958	0.958	0.955
30	20	20	18.69	19.18	19.31	19.13	0.960	0.957	0.943
50	20	20	18.41	18.91	18.94	18.88	0.959	0.958	0.948
100	20	20	18.15	18.66	18.64	18.63	0.958	0.958	0.952
200	20	20	18.01	18.50	18.47	18.47	0.958	0.958	0.955
				$\alpha = 0.01$					
13	5	5	24.97	25.25	25.98	25.60	0.993	0.992	0.975
15	5	5	24.79	25.05	25.48	25.22	0.992	0.991	0.977
20	5	5	24.24	24.53	24.68	24.56	0.991	0.991	0.980
25	5	5	23.88	24.19	24.20	24.13	0.991	0.991	0.982
13	10	10	24.30	24.71	25.56	24.79	0.993	0.992	0.980
15	10	10	24.17	24.41	25.12	24.56	0.993	0.991	0.980
20	10	10	23.89	24.18	24.42	24.13	0.992	0.991	0.982
25	10	10	23.57	23.97	24.01	23.84	0.991	0.991	0.984
30	10	10	23.44	23.80	23.73	23.62	0.991	0.991	0.984
50	10	10	22.96	23.15	23.16	23.13	0.991	0.991	0.987
100	10	10	22.48	22.72	22.71	22.70	0.991	0.991	0.989
200	10	10	22.19	22.44	22.46	22.46	0.991	0.991	0.990
30	20	20	23.13	23.33	23.57	23.33	0.991	0.991	0.986
50	20	20	22.79	22.95	23.07	23.00	0.991	0.991	0.987
100	20	20	22.46	22.61	22.67	22.66	0.991	0.991	0.989
200	20	20	22.25	22.43	22.45	22.45	0.991	0.991	0.990

Note. $\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha_c))$, $\widetilde{\text{CP}}_{\text{YS}\cdot F} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha_c))$, $\widetilde{\text{CP}}_{\chi^2} = \widetilde{\text{CP}}(\chi^2_{p,\alpha_c})$, $\chi^2_{6,\alpha_c} = 18.29$ $(\alpha = 0.05)$, $\chi^2_{6,\alpha_c} = 22.21$ $(\alpha = 0.01)$, $\alpha_c = \alpha/(m-1)$.

Table 12: Simulated and approximate values and coverage probabilities for comparisons with a control ($m = 10$ and $(p_1, p_2, p_3) = (12, 8, 4)$)

Sample Size			Upper Percentile				Coverage Probability		
$n_1^{(\ell)}$	$n_2^{(\ell)}$	$n_3^{(\ell)}$	$\widetilde{t}^2_{\text{simu}\cdot\text{c}}$	$\widetilde{t}^2_{\text{simu}\cdot\text{Bon}}$	$t^2_{\text{YS}\cdot\text{L}}$	$t^2_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}}$	$\widetilde{\text{CP}}_{\text{YS}\cdot F}$	$\widetilde{\text{CP}}_{\chi^2}$
				$\alpha = 0.05$					
13	5	5	29.95	30.70	34.19	33.52	0.983	0.980	0.919
15	5	5	30.51	31.09	33.33	32.87	0.976	0.973	0.907
20	5	5	30.56	31.18	31.98	31.77	0.966	0.964	0.904
25	5	5	30.21	30.81	31.19	31.08	0.962	0.960	0.911
13	10	10	28.20	28.82	33.50	32.15	0.989	0.983	0.947
15	10	10	29.05	29.74	32.74	31.77	0.982	0.976	0.934
20	10	10	29.69	30.32	31.56	31.08	0.970	0.966	0.922
25	10	10	29.71	30.31	30.88	30.60	0.964	0.961	0.921
30	10	10	29.51	30.05	30.42	30.25	0.961	0.959	0.924
50	10	10	28.85	29.41	29.50	29.46	0.959	0.958	0.936
100	10	10	28.28	28.79	28.78	28.77	0.957	0.957	0.946
200	10	10	27.87	28.34	28.39	28.39	0.957	0.957	0.952
30	20	20	29.08	29.62	30.16	29.77	0.963	0.959	0.932
50	20	20	28.70	29.31	29.36	29.24	0.959	0.957	0.939
100	20	20	28.19	28.66	28.73	28.70	0.957	0.957	0.947
200	20	20	27.85	28.40	28.38	28.37	0.957	0.957	0.952
				$\alpha = 0.01$					
13	5	5	36.03	36.55	40.51	39.64	0.997	0.996	0.975
15	5	5	36.54	36.78	39.40	38.81	0.995	0.995	0.971
20	5	5	36.44	36.65	37.67	37.41	0.993	0.992	0.971
25	5	5	35.96	36.23	36.66	36.52	0.992	0.992	0.974
13	10	10	33.87	34.17	39.63	37.88	0.998	0.997	0.986
15	10	10	34.76	35.27	38.65	37.41	0.997	0.995	0.981
20	10	10	35.39	35.78	37.14	36.52	0.994	0.993	0.978
25	10	10	35.31	35.64	36.27	35.91	0.993	0.992	0.978
30	10	10	34.98	35.25	35.69	35.46	0.992	0.991	0.980
50	10	10	34.20	34.42	34.52	34.46	0.991	0.991	0.984
100	10	10	33.42	33.63	33.61	33.60	0.991	0.991	0.987
200	10	10	32.88	33.10	33.13	33.12	0.991	0.991	0.989
30	20	20	34.48	34.67	35.35	34.86	0.992	0.991	0.982
50	20	20	33.91	34.09	34.34	34.19	0.991	0.991	0.985
100	20	20	33.33	33.52	33.54	33.52	0.991	0.991	0.987
200	20	20	32.87	33.29	33.11	33.10	0.991	0.991	0.989

Note. $\widetilde{\text{CP}}_{\text{YS}\cdot\text{L}} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot\text{L}}(\alpha_\text{c}))$, $\widetilde{\text{CP}}_{\text{YS}\cdot F} = \widetilde{\text{CP}}(t^2_{\text{YS}\cdot F}(\alpha_\text{c}))$, $\widetilde{\text{CP}}_{\chi^2} = \widetilde{\text{CP}}(\chi^2_{p,\alpha_\text{c}})$, $\chi^2_{12,\alpha_\text{c}} = 27.99$ ($\alpha = 0.05$), $\chi^2_{12,\alpha_\text{c}} = 32.62$ ($\alpha = 0.01$), $\alpha_\text{c} = \alpha/(m-1)$.

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