Likelihood ratio test statistic for block compound symmetry covariance structure and its asymptotic expansion

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Abstract

This paper is concerned with a test for blocked compound symmetry (BCS) covariance structure under normality. BCS structure is an extension of intraclass covariance structure. We derive two asymptotic expansions of the null distribution of the likelihood ratio statistic. One is an asymptotic expansion in terms of χ^2 distributions under a classical large sample framework. The other is a high-dimensional Edgeworth expansion when the number of variable q, u and the sample size N approach ∞ together, while the ratio qu/N is converging on a finite nonzero limit $c \in (0, 1)$. Finally, numerical simulations reveal that the accuracy of our asymptotic expansions.

Key Words and Phrases: Blocked compound symmetric structure; Likelihood ratio test; Asymptotic expansion.

1 Introduction

Suppose that a variable vector $\boldsymbol{x}^* = (x_1, x_2, \ldots, x_q)'$ is measured at u points $t_1, t_2, \ldots t_u$, and let the variable vector $\boldsymbol{x}^*$ measured at the t_i time point be denoted by $\boldsymbol{x}_i^*$. The q-vectors $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \ldots, \boldsymbol{x}_u^*$ are equally correlated if they have the following equicorrelated covariance:

$$\mathrm{Cov}(\boldsymbol{x}_r^*, \boldsymbol{x}_s^*) = \begin{cases} \boldsymbol{\Sigma}_0 & (r = s) \\ \boldsymbol{\Sigma}_1 & (r \neq s), \end{cases}$$

where $\boldsymbol{\Sigma}_0$ is a $q \times q$ positive definite symmetric matrix, $\boldsymbol{\Sigma}_1$ is a $q \times q$ symmetric matrix and $r, s = 1, 2, \ldots, u$. Further, suppose that there are random samples of $\boldsymbol{x} = (\boldsymbol{x}_1^{*\prime}, \boldsymbol{x}_2^{*\prime}, \ldots, \boldsymbol{x}_u^{*\prime})'$, and let the random samples be denoted by $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$, which are independently distributed as $\mathcal{N}_{qu}(\boldsymbol{\mu}, \boldsymbol{\Gamma})$ which has the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Gamma}$. The covariance matrix $\boldsymbol{\Gamma}$ has the structure as follows:

$$\boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{\Sigma}_0 & \boldsymbol{\Sigma}_1 & \cdots & \boldsymbol{\Sigma}_1 \\ \boldsymbol{\Sigma}_1 & \boldsymbol{\Sigma}_0 & \cdots & \boldsymbol{\Sigma}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Sigma}_1 & \boldsymbol{\Sigma}_1 & \cdots & \boldsymbol{\Sigma}_0 \end{pmatrix} = \boldsymbol{I}_u \otimes (\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1) + \boldsymbol{J}_u \otimes \boldsymbol{\Sigma}_1, \tag{1.1}$$

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where $\boldsymbol{I}_u$ is the $u \times u$ identity matrix, $\mathbf{1}_u = (1, \ldots, 1) \in \mathbb{R}^u$, and $\boldsymbol{J}_u = \mathbf{1}_u \mathbf{1}_u'$. The covariance structure is called "blocked compound symmetric (BCS)" structure. If $q = 1$, namely, $\boldsymbol{\Sigma}_0 = \sigma_0 = \sigma$ and $\boldsymbol{\Sigma}_1 = \sigma_1 = \tau\sigma$, where $-1/(u-1) \leq \tau \leq 1$, $\boldsymbol{\Gamma} = \sigma\left\{(1-\tau)\boldsymbol{I}_u + \tau\boldsymbol{J}_u\right\}$. This covariance structure is called intraclass correlation structure. That is, BCS structure is on extension of intraclass covariance structure. Intraclass correlation structure is an important covariance structure for repeated measures data. BCS structure is also an important covariance structure too.

Leiva (2007) discussed the maximum likelihood estimator and the likelihood ratio test statistic for testing of BCS structure with structured mean vector. Roy and Leiva (2011) considered the χ^2 approximation of the likelihood ratio test statistic. Srivastava et al. (2008) referred to an estimate for BCS structure. Also, a linear discrimination method was developed to be used when the training vectors have a BCS covariance structure in Leiva (2007).

In this paper, we derive two asymptotic expansions of the null distribution of the likelihood ratio criterion. One is an asymptotic expansion in terms of χ^2 distributions under a classical large sample framework A1 : q, u are fixed, $N \to \infty$. In general, χ^2 approximation has good accuracies when the number of variables is small. But under BCS structure, the number of variables qu tends to get larger, namely this accuracy gets worse as qu gets larger. As an approach to overcoming this fault, it has been attempted to derive high-dimensional approximations under a high-dimensional framework A2 : $q, u, N \to \infty$, $qu/N \to c \in (0, 1)$. Kato et al. (2010) derived a high-dimensional asymptotic expansion of likelihood ratio criterion for testing intraclass correlation structure and its error bound.

The following section is organized as follows: in Section 2 we introduce the likelihood ratio statistic and calculate the hth moment of the likelihood ratio statistic under null hypothesis, in Section 3 we derive the coefficient of Bartlett correction and an asymptotic expansion of the null distribution under the large sample framework A1, in Section 4 we derive a high-dimensional asymptotic expansion of the null distribution under the high-dimensional framework A2, in Section 5 we investigate the accuracy of presented asymptotic expansions and the others by Monte Carlo simulation. Some preliminary results and the order of cumulant, which is needed to guarantee the derivation of hth moment and the asymptotic expansions of the likelihood ratio statistics are given in Appendix.

2 Likelihood ratio test statistic for BCS structure hypothesis

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be a random sample from qu-population $\mathcal{N}_{qu}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Gamma}$ is defined in (1.1). We assume that $\boldsymbol{\Sigma}_0 + (u-1)\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1$ are positive definite matrices. The inverse and determinant for $\boldsymbol{\Gamma}$ can be written as

$$\begin{aligned}
\boldsymbol{\Gamma}^{-1} &= \boldsymbol{I}_u \otimes \boldsymbol{\Delta}_1^{-1} + \boldsymbol{J}_u \otimes \frac{1}{u}\left(\boldsymbol{\Delta}_0^{-1} - \boldsymbol{\Delta}_1^{-1}\right), \\
|\boldsymbol{\Gamma}| &= |\boldsymbol{\Delta}_0||\boldsymbol{\Delta}_1|^{u-1},
\end{aligned}$$

respectively, where

$$\begin{aligned}
\boldsymbol{\Delta}_0 &= \boldsymbol{\Sigma}_0 + (u-1)\boldsymbol{\Sigma}_1, \\
\boldsymbol{\Delta}_1 &= \boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1.
\end{aligned}$$

For the proof, see Leiva (2007). Let $\boldsymbol{x}_i = (\boldsymbol{x}_{i1}', \boldsymbol{x}_{i2}', \dots \boldsymbol{x}_{iu}')'$ and $\boldsymbol{x}_{ij}$ is $q \times 1$, $i = 1, 2, \dots, N$, $j = 1, 2, \dots, u$. We can express the maximum likelihood estimators (MLEs) of $\boldsymbol{\mu}$ and $\boldsymbol{\Gamma}$ as follows:

$$\begin{aligned}
\widehat{\boldsymbol{\mu}} &= \bar{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_i \\
&= (\bar{\boldsymbol{x}}_1', \bar{\boldsymbol{x}}_2', \dots, \bar{\boldsymbol{x}}_u')' = \left(\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_{i1}', \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_{i2}', \dots, \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_{iu}' \right)', \qquad (2.1) \\
\widehat{\boldsymbol{\Gamma}} &= \boldsymbol{I}_u \otimes (\widehat{\boldsymbol{\Sigma}}_0 \widehat{\boldsymbol{\Sigma}}_1) + \boldsymbol{J}_u \otimes \widehat{\boldsymbol{\Sigma}}_1 \\
&= \boldsymbol{I}_u \otimes \frac{1}{Nu}\left(\boldsymbol{C}_0 - \frac{1}{(u-1)}\boldsymbol{C}_1 \right) + \boldsymbol{J}_u \otimes \frac{1}{Nu(u-1)}\boldsymbol{C}_1, \qquad (2.2)
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{V}_{rs} &= \sum_{i=1}^{N} (\boldsymbol{x}_{ir} - \bar{\boldsymbol{x}}_r)(\boldsymbol{x}_{is} - \bar{\boldsymbol{x}}_s)', \\
\boldsymbol{C}_0 &= \sum_{r=1}^{u} \boldsymbol{V}_{rr}, \quad \boldsymbol{C}_1 = \sum\sum_{r \neq s} \boldsymbol{V}_{rs}.
\end{aligned}$$

For the proof, see Appendix. Moreover, we can express the MLEs of $\boldsymbol{\Delta}_0$ and $\boldsymbol{\Delta}_1$ as follows:

$$\begin{aligned}
\widehat{\boldsymbol{\Delta}}_0 &= \frac{1}{Nu}\sum_{r=1}^{u}\sum_{s=1}^{u} \boldsymbol{V}_{rs}, \\
\widehat{\boldsymbol{\Delta}}_1 &= \frac{1}{Nu}\sum_{r=1}^{u} \boldsymbol{V}_{rr} - \frac{1}{Nu(u-1)}\sum\sum_{r \neq s} \boldsymbol{V}_{rs}.
\end{aligned}$$

The likelihood ratio criterion for testing the hypothesis

$$H_0 : \boldsymbol{\Sigma} = \boldsymbol{I}_u \otimes (\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1) + \boldsymbol{J}_u \otimes \boldsymbol{\Sigma}_1 (= \boldsymbol{\Gamma})$$

against the alternatives that H_0 is not true, is given by

$$\begin{aligned}
\lambda &= \frac{|\boldsymbol{S}|^{\frac{n}{2}}}{|\boldsymbol{I}_u \otimes (\widehat{\boldsymbol{\Sigma}}_0 - \widehat{\boldsymbol{\Sigma}}_1) + \boldsymbol{J}_u \otimes \widehat{\boldsymbol{\Sigma}}_1|^{\frac{n}{2}}} \\
&= \frac{|\boldsymbol{V}|^{\frac{n}{2}}}{|N\widehat{\boldsymbol{\Delta}}_0|^{\frac{n}{2}} |N\widehat{\boldsymbol{\Delta}}_1|^{\frac{n(u-1)}{2}}} \\
&= \frac{(u-1)^{\frac{nq(u-1)}{2}} |\boldsymbol{V}|^{\frac{n}{2}}}{|N\widehat{\boldsymbol{\Delta}}_0|^{\frac{n}{2}} |N(u-1)\widehat{\boldsymbol{\Delta}}_1|^{\frac{n(u-1)}{2}}}, \qquad (2.3)
\end{aligned}$$

where $\boldsymbol{V} = N\boldsymbol{S} = \sum_{i=1}^{N} (\boldsymbol{x}_i - \bar{\boldsymbol{x}})(\boldsymbol{x}_i - \bar{\boldsymbol{x}})'$ and $n = N - 1$. From $\boldsymbol{V} \sim \mathcal{W}_{qu}(n, \boldsymbol{\Gamma})$, we have the following result.

Theorem 1. *$\boldsymbol{V} \sim \mathcal{W}_{qu}(n, \boldsymbol{\Gamma})$, then $\widehat{\boldsymbol{\Delta}}_0$ and $\widehat{\boldsymbol{\Delta}}_1$ are distributed independently and*

$$N\widehat{\boldsymbol{\Delta}}_0 \sim \mathcal{W}_q(n, \boldsymbol{\Delta}_0), \tag{2.4}$$

$$N(u-1)\widehat{\boldsymbol{\Delta}}_1 \sim \mathcal{W}_q(n(u-1), \boldsymbol{\Delta}_1). \tag{2.5}$$

For the proof, see Appendix. Then, using a distributional result (see, e.g. Muirhead (1982) and Fujikoshi et al. (2010)) that

$$\frac{|N\widehat{\boldsymbol{\Delta}}_0|}{|\boldsymbol{\Sigma}_0 + (u-1)\boldsymbol{\Sigma}_1|} \sim \prod_{r=1}^{q} \chi^2_{n-r+1}, \tag{2.6}$$

$$\frac{|N(u-1)\widehat{\boldsymbol{\Delta}}_1|}{|\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1|} \sim \prod_{r=1}^{q} \chi^2_{n(u-1)-r+1}, \tag{2.7}$$

where for $r = 1, 2, \ldots, q$, χ^2_{n-r+1} and $\chi^2_{n(u-1)-r+1}$ denote mutually independent random variables, each following the χ^2 distribution with $n-r+1$ and $n(u-1)-r+1$ degree of freedom respectively. Moreover, we can express the expectations of (2.6) and (2.7) as follow:

$$\begin{aligned} \mathrm{E}\left[\left(\frac{|N\widehat{\boldsymbol{\Delta}}_0|}{|\boldsymbol{\Sigma}_0 + (u-1)\boldsymbol{\Sigma}_1|}\right)^{-h}\right] &= 2^{-hq} \prod_{r=1}^{q} \frac{\Gamma\left[\frac{n-r+1}{2} - h\right]}{\Gamma\left[\frac{n-r+1}{2}\right]} \\ &= 2^{-hq} \frac{\Gamma_q\left[\frac{n-2h}{2}\right]}{\Gamma_q\left[\frac{n}{2}\right]}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \mathrm{E}\left[\left(\frac{|N(u-1)\widehat{\boldsymbol{\Delta}}_1|}{|\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1|}\right)^{-h(u-1)}\right] &= 2^{-hq(u-1)} \prod_{r=1}^{q} \frac{\Gamma\left[\frac{n(u-1)-r+1}{2} - h(u-1)\right]}{\Gamma\left[\frac{n(u-1)-r+1}{2}\right]} \\ &= 2^{-hq(u-1)} \frac{\Gamma_q\left[\frac{(n-2h)(u-1)}{2}\right]}{\Gamma_q\left[\frac{n(u-1)}{2}\right]}, \end{aligned} \tag{2.9}$$

where

$$\Gamma_p\left[a\right] = \pi^{\frac{p(p-1)}{4}} \prod_{r=1}^{p} \Gamma\left[a - \frac{1}{2}(r-1)\right].$$

Let $\Lambda = \lambda^{\frac{2}{n}}$. From (2.8), (2.9), we find the hth moment of Λ as follows:

$$\begin{aligned}
\mathrm{E}\left[\Lambda^h\right] &= \int \left\{ \frac{(u-1)^{q(u-1)}|\boldsymbol{V}|}{|N\widehat{\boldsymbol{\Delta}}_0||N(u-1)\widehat{\boldsymbol{\Delta}}_1|^{(u-1)}} \right\}^h \frac{|\boldsymbol{V}|^{\frac{n-qu-1}{2}}}{2^{\frac{nqu}{2}}\Gamma_{qu}\left[\frac{n}{2}\right]|\boldsymbol{\Gamma}|^{\frac{n}{2}}} \mathrm{etr}\left(-\frac{1}{2}\boldsymbol{\Gamma}^{-1}\boldsymbol{V}\right)\mathrm{d}\boldsymbol{V} \\
&= 2^{hqu}(u-1)^{hq(u-1)}\frac{\Gamma_{qu}\left[\frac{n+2h}{2}\right]}{\Gamma_{qu}\left[\frac{n}{2}\right]}|\boldsymbol{\Gamma}|^h \int |N\widehat{\boldsymbol{\Delta}}_0|^{-h}|N(u-1)\widehat{\boldsymbol{\Delta}}_1|^{-h(u-1)} \\
&\quad \times \frac{|\boldsymbol{V}|^{\frac{n+2h-qu-1}{2}}}{2^{\frac{(n+2h)qu}{2}}\Gamma_{qu}\left[\frac{n+2h}{2}\right]|\boldsymbol{\Gamma}|^{\frac{n+2h}{2}}} \mathrm{etr}\left(-\frac{1}{2}\boldsymbol{\Gamma}^{-1}\boldsymbol{V}\right)\mathrm{d}\boldsymbol{V} \\
&= 2^{hqu}(u-1)^{hq(u-1)}\frac{\Gamma_{qu}\left[\frac{n+2h}{2}\right]}{\Gamma_{qu}\left[\frac{n}{2}\right]}\mathrm{E}\left[\left(\frac{|N\widehat{\boldsymbol{\Delta}}_0|}{|\boldsymbol{\Sigma}_0+(u-1)\boldsymbol{\Sigma}_1|}\right)^{-h}\right]\mathrm{E}\left[\left(\frac{|N(u-1)\widehat{\boldsymbol{\Delta}}_1|}{|\boldsymbol{\Sigma}_0-\boldsymbol{\Sigma}_1|}\right)^{-h(u-1)}\right] \\
&= (u-1)^{hq(u-1)}\frac{\Gamma_{qu}\left[\frac{n+2h}{2}\right]}{\Gamma_{qu}\left[\frac{n}{2}\right]}\frac{\Gamma_q\left[\frac{n(u-1)}{2}\right]}{\Gamma_q\left[\frac{(n+2h)(u-1)}{2}\right]}\frac{\Gamma_q\left[\frac{n}{2}\right]}{\Gamma_q\left[\frac{n+2h}{2}\right]}. \qquad (2.10)
\end{aligned}$$

If $q = 1$, λ in (2.3) and hth moment of Λ in (2.10) coincide with the intraclass correlation's respectively.

3 Asymptotic expansion of the null distribution of λ under a large-sample framework

First we consider asymptotic expansion of the null distribution of $-2\log\lambda$ under a large-sample framework:

$$\mathrm{A1} : q, u \text{ are fixed}, \; N \to \infty.$$

From (2.10), the characteristic function of $-2\log\lambda$ is written as

$$C(t) = \mathrm{E}\left[\exp\{-2it\log\lambda\}\right] = \mathrm{E}\left[\lambda^{-2it}\right] = \mathrm{E}\left[\Lambda^{-int}\right]$$

and the logarithm of $C(t)$ is

$$\begin{aligned}
\log C(t) = -itnq(u-1)\log(u-1) + \log\Gamma_{qu}\left[\frac{n}{2}(1-2it)\right] - \log\Gamma_q\left[\frac{n(u-1)}{2}(1-2it)\right] \\
-\log\Gamma_q\left[\frac{n}{2}(1-2it)\right] - \log\Gamma_{qu}\left[\frac{n}{2}\right] + \log\Gamma_q\left[\frac{n(u-1)}{2}\right] + \log\Gamma_q\left[\frac{n}{2}\right]. \qquad (3.1)
\end{aligned}$$

To obtain an expansion for $\log C(t)$, we use the following generalized version of Stirling's formula for the gamma function (see, e.g. Muirhead (1982)):

$$\log\Gamma(a+z) = \frac{1}{2}\log(2\pi) + \left(a+z-\frac{1}{2}\right)\log a - a + \frac{B_2(z)}{2a} - \frac{B_3(z)}{6a^2} + O\left(\frac{1}{a^3}\right), \qquad (3.2)$$

where $B_2(z) = z^2 - z + 1/6$, $B_3(z) = z^3 - (3/2)z^2 + (1/2)z$.

Expanding each of the gamma functions in (3.1), we can express the $\log C(t)$ as follows:

$$\log C(t) = \log(1-2it)^{-\frac{1}{2}f} + \frac{1}{n}\beta_1(m-1) + \frac{1}{n^2}\beta_2(m^2-1) + O(n^{-3}),$$

where

$$\begin{aligned}
\beta_1 &= v_1 - \frac{u}{u-1}v_2, \\
\beta_2 &= \frac{2}{3}\left\{w_1 - \frac{(u-1)^2+1}{(u-1)^2}w_2\right\}, \\
f &= \frac{qu(qu+1)}{2} - q(q+1), \\
m &= (1-2it)^{-1},
\end{aligned}$$

and

$$\begin{aligned}
v_1 &= \frac{qu}{24}(2q^2u^2+3qu-1), \\
v_2 &= \frac{q}{24}(2q^2+3q-1), \\
w_1 &= -\frac{1}{32}qu(qu-1)(qu+1)(qu+2), \\
w_2 &= -\frac{1}{32}q(q-1)(q+1)(q+2),
\end{aligned}$$

respectively. From these findings, $C(t)$ can be expanded as

$$C(t) = (1-2it)^{-\frac{1}{2}f}\left[1 + \frac{1}{n}\beta_1(m-1) + \frac{1}{n^2}\left\{\beta_2(m^2-1) + \beta_1^2(m-1)^2\right\}\right] + O(n^{-3}). \tag{3.3}$$

From (3.3), we can get the coefficient of Bartlett correction as follows:

$$\gamma = \frac{u(2q^2u^3 - 2q^2u^2 + 3qu^2 - 3qu - u - 2q^2 - 3q + 2)}{6(u-1)(qu^2+u-2q-2)}. \tag{3.4}$$

Therefore, we obtain the following result.

Theorem 2. *Let λ be the LR criterion for testing BCS structure* (1.1) *given in* (2.3). *Then,*

$$\Pr(-2\rho\log\lambda \le x) = G_f(x) + \frac{\omega}{M^2}\left\{G_{f+4}(x) - G_f(x)\right\} + O(n^{-3}),$$

where G_f is the distribution function of the χ^2 distribution with f degree of freedom, $\rho = 1-\gamma/n$, γ is the coefficient of Bartlett correction given (3.4), *$M = n\rho$ and*

$$\begin{aligned}
\omega = \frac{1}{288}\frac{q}{(u-1)^2(qu^2+u-2q-2)}&\left(2q^4u^8 - 4q^4u^7 + 6q^3u^7 - 10q^4u^6 - 24q^3u^6 + q^2u^6\right. \\
&+ 32q^4u^5 + 18q^3u^5 - 30q^2u^5 - 12qu^5 - 26q^4u^4 + 24q^3u^4 + 89q^2u^4 + 42qu^4 - 13u^4 \\
&+ 12q^4u^3 - 18q^3u^3 - 94q^2u^3 - 30qu^3 + 64u^3 - 4q^4u^2 + 12q^3u^2 + 59q^2u^2 - 48qu^2 \\
&\left.- 112u^2 - 24q^4u - 84q^3u - 48q^2u + 108qu + 96u + 24q^4 + 72q^3 + 24q^2 - 72q - 48\right).
\end{aligned}$$

If $q=1$, this result coincides with the result of testing intraclass correlation structure.

4 Asymptotic expansion of the null distribution of λ under a high-dimensional framework

In general, an asymptotic expansion in terms of χ^2 distributions under a large sample framework gets worse as the number of variables qu gets larger. It seems the data vectors which test the BCS structure tends to be large number of variables. Accordingly, it has been attempted to derive high-dimensional approximations of the null distribution of λ under a high-dimensional framework:

$$\mathrm{A2}: q, u, N \to \infty,\ \frac{qu}{N} \to c \in (0,1).$$

The hth moment of Λ is defined in (2.10). Hence, the characteristic function of $T = -\log\Lambda$ is given by

$$\begin{aligned} C_T(t) &= \mathrm{E}\left[\exp\{itT\}\right] = \mathrm{E}\left[\Lambda^{-it}\right] \\ &= (u-1)^{-itq(u-1)} \frac{\Gamma_{qu}\left[\frac{n}{2} - it\right]}{\Gamma_{qu}\left[\frac{n}{2}\right]} \frac{\Gamma_q\left[\frac{n(u-1)}{2}\right]}{\Gamma_q\left[\frac{n(u-1)}{2} - it(u-1)\right]} \frac{\Gamma_q\left[\frac{n}{2}\right]}{\Gamma_q\left[\frac{n}{2} - it\right]} \\ &= K(u-1)^{-itq(u-1)} \frac{\prod_{j=q}^{qu-1} \Gamma\left[-it + \frac{1}{2}(n-j)\right]}{\prod_{j=0}^{q-1} \Gamma\left[-it(u-1) + \frac{1}{2}\{n(u-1)-j\}\right]}, \end{aligned}$$

where $K = \prod_{j=0}^{q-1}\Gamma\left[\frac{1}{2}\{n(u-1)-j\}\right] / \prod_{j=q}^{qu-1}\Gamma\left[\frac{1}{2}(n-j)\right]$. Accordingly, the cumulant generating function of T can be expressed as

$$\begin{aligned} \log C_T(t) = \log K - itq(u-1)\log(u-1) &+ \sum_{j=q}^{qu-1} \log\Gamma\left[-it + \frac{1}{2}(n-j)\right] \\ &- \sum_{j=0}^{q-1} \log\Gamma\left[-it(u-1) + \frac{1}{2}\{n(u-1)-j\}\right]. \end{aligned}$$

Now we use the Taylor expansion formula:

$$\log\Gamma(a+z) = \log\Gamma(a) + \sum_{k=1}^{\infty} \frac{1}{k!}\psi^{(k-1)}(a) z^k, \tag{4.1}$$

where ψ is the di-gamma function defined by $\psi(v) = (\mathrm{d}/\mathrm{d}v)\log\Gamma(v)$. It is known that the polygamma function $\psi^{(s)}(a) = (\mathrm{d}^s/\mathrm{d}v^s)\psi(v)|_{v=a}$ can be expressed as

$$\psi^{(s)}(a) = \begin{cases} -C + \displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{1+k} - \frac{1}{k+a}\right), & s = 0, \\ \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{s+1}s!}{(k+a)^{s+1}}, & s = 1, 2, \ldots, \end{cases}$$

where C is the Euler constant. Then, we can write

$$
\begin{aligned}
\log C_T(t) &= \log K - itq(u-1)\log(u-1) \\
&\quad + \sum_{j=q}^{qu-1}\left\{\log\Gamma\left[\frac{1}{2}(n-j)\right] + \sum_{k=1}^{\infty}(-1)^k\frac{(it)^k}{k!}\psi^{(k-1)}\left(\frac{1}{2}(n-j)\right)\right\} \\
&\quad - \sum_{j=0}^{q-1}\left\{\log\Gamma\left[\frac{1}{2}\left(n(u-1)-j\right)\right] + \sum_{k=1}^{\infty}(-1)^k\frac{(it)^k}{k!}\psi^{(k-1)}\left(\frac{1}{2}(n(u-1)-j)\right)(u-1)^k\right\} \\
&= \sum_{r=1}^{\infty}\kappa^{(r)}\frac{(it)^r}{r!} \\
&= it\mu + \frac{1}{2}(it)^2\sigma^2 + \sum_{r=3}^{\infty}\kappa^{(r)}\frac{(it)^r}{r!},
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
\mu &= -q(u-1)\log(u-1) - \sum_{j=q}^{qu-1}\psi^{(0)}\left(\frac{1}{2}(n-j)\right) + \sum_{j=0}^{q-1}\psi^{(0)}\left(\frac{1}{2}(n(u-1)-j)\right)(u-1), \\
\sigma^2 &= \sum_{j=q}^{qu-1}\psi^{(1)}\left(\frac{1}{2}(n-j)\right) - \sum_{j=0}^{q-1}\psi^{(1)}\left(\frac{1}{2}(n(u-1)-j)\right)(u-1)^2, \qquad (4.3) \\
\kappa^{(r)} &= (-1)^r\left\{\sum_{j=q}^{qu-1}\psi^{(r-1)}\left(\frac{1}{2}(n-j)\right) - \sum_{j=0}^{q-1}\psi^{(r-1)}\left(\frac{1}{2}(n(u-1)-j)\right)(u-1)^r\right\},\ r \geq 3.
\end{aligned}
$$

From A.3 in Appendix, we find that $\sigma^2 = \mathrm{O}_0^*$ and $\kappa^{(\ell)} = \mathrm{O}_{\ell-2}^*$, where O_j^* denotes a term of jth order with respect to (n^{-1}, qu^{-1}).

Let

$$
Z = \frac{T-\mu}{\sqrt{\sigma^2}} = \frac{-(2/n)\log\lambda - \mu}{\sqrt{\sigma^2}}. \tag{4.4}
$$

Then, using (6.1), characteristic function of Z can be expressed as follows:

$$
\begin{aligned}
C_Z(t) &= \mathrm{E}\left[\exp(itZ)\right] \\
&= \exp\left\{\log C_T\left(\frac{t}{\sqrt{\sigma^2}}\right) - i\mu\frac{t}{\sqrt{\sigma^2}}\right\} \\
&= \exp\left(-\frac{t^2}{2} + \sum_{j=3}^{\infty}\frac{\kappa^{(j)}}{j!\left(\sqrt{\sigma^2}\right)^j}(it)^j\right) \\
&= \exp\left(-\frac{t^2}{2}\right)\left\{1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(\sum_{j=3}^{\infty}\frac{\kappa^{(j)}}{j!\left(\sqrt{\sigma^2}\right)^j}(it)^j\right)^k\right\} \\
&= \exp\left(-\frac{t^2}{2}\right)\left\{1 + \sum_{k=1}^{\infty}\frac{1}{k!}(it)^{3k}\left(\sum_{j=0}^{\infty}\frac{\tilde{\kappa}^{(j+3)}}{(j+3)!}(it)^j\right)^k\right\} \\
&= \exp\left(-\frac{t^2}{2}\right)\left\{1 + \sum_{k=1}^{\infty}\frac{1}{k!}(it)^{3k}\sum_{j=0}^{\infty}\gamma_{k,j}(it)^j\right\},
\end{aligned}
$$

where $\tilde{\kappa}^{(j)} = \kappa^{(j)}/(\sqrt{\sigma^2})^j$ and

$$
\gamma_{k,j} = \sum_{j_1+\cdots+j_k=j}\prod_{l=1}^{k}\frac{\tilde{\kappa}^{(j_l+3)}}{(j_l+3)!}. \tag{4.5}
$$

Let

$$
C_Z^{(s)}(t) = \exp\left(-\frac{t^2}{2}\right)\left\{1 + \sum_{k=1}^{s}\frac{1}{k!}(it)^{3k}\sum_{j=0}^{s-k}\gamma_{k,j}(it)^j\right\}. \tag{4.6}
$$

Then, we have

$$
C_Z(t) - C_Z^{(s)}(t) = \exp\left(-\frac{t^2}{2}\right)\left\{\sum_{k=1}^{s}\frac{1}{k!}(it)^{3k}\sum_{j=s-k+1}^{\infty}\gamma_{k,j}(it)^j + \sum_{k=s+1}^{\infty}\frac{1}{k!}\left(\sum_{j=3}^{\infty}\frac{\tilde{\kappa}^{(j)}}{j!}(it)^j\right)^k\right\}.
$$

Inverting (4.6) formally, we obtain asymptotic expansion of the distribution:

$$
\Phi_s(x) = \Phi(x) - \phi(x)\left\{\sum_{k=1}^{s}\sum_{j=0}^{s-k}\gamma_{k,j}h_{3k+j-1}(x)\right\}, \tag{4.7}
$$

where Φ and ϕ are the distribution function of the standard normal distribution and its density function, respectively; $\gamma_{k,j}$ is given by (4.5); $h_r(x)$ denotes the rth order Hermite polynomial defined by

$$
\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^r \exp\left(-\frac{1}{2}x^2\right) = (-1)^r h_r(x)\exp\left(-\frac{1}{2}x^2\right).
$$

Theorem 3. *Let* λ *be the LR criterion for testing BCS structure* (1.1) *given in* (2.3)*. Let* $Z = \{-(2/n)\log\lambda - \mu\}/\sqrt{\sigma^2}$ *be the standardized statistic in* (4.4)*. Then,*

$$P(Z \le x) = \Phi_s(x) + \mathrm{O}^*_{s+1}, \tag{4.8}$$

where $\Phi_s(x)$ *is given by* (4.7).

As the special cases of (4.8),

$$\Phi_0(x) = \Phi(x), \tag{4.9}$$

$$\Phi_1(x) = \Phi(x) - \frac{1}{6}\phi(x)\tilde{\kappa}^{(3)}(x^2-1), \tag{4.10}$$

$$\Phi_2(x) = \Phi(x) - \phi(x)\left\{\frac{1}{6}\tilde{\kappa}^{(3)}(x^2-1) + \frac{1}{24}\tilde{\kappa}^{(4)}(x^3-2x) + \frac{1}{36}(\tilde{\kappa}^{(3)})^2(x^5-10x^3+15x)\right\}. \tag{4.11}$$

5 Numerical comparison

In this section we present the simulations results under various setting of q, u and N in order to investigate the accuracy of proposed asymptotic expansions. We take

$$\boldsymbol{\Sigma}_0 = \begin{pmatrix} 2.0 & 1.0 & \cdots & 1.0 \\ 1.0 & 2.0 & \cdots & 1.0 \\ \vdots & \vdots & \ddots & \vdots \\ 1.0 & 1.0 & \cdots & 2.0 \end{pmatrix}, \boldsymbol{\Sigma}_1 = \begin{pmatrix} 1.0 & 0.5 & \cdots & 0.5 \\ 0.5 & 1.0 & \cdots & 0.5 \\ \vdots & \vdots & \ddots & \vdots \\ 0.5 & 0.5 & \cdots & 1.0 \end{pmatrix}.$$

We list the estimated significance levels of $-2\rho\log\lambda$ using the χ^2 distribution in Table 1, Table 2, Table 3 and of Z using the standard normal distribution in Table 4, Table 5, Table 6 for $N = n+1 = 20$ and 100 calculated by using $1,000,000$ repetitions with nominal significance levels of 0.01, 0.05, 0.50, 0.95, 0.99. In order to get these values, we need to calculate $\psi^{(r)}(a)$, which is an infinite series. It can be calculated numerically with suitable precision by using software, e.g., R. From the tables, we can see that the χ^2 approximation and an asymptotic expansion of $-2\rho\log\lambda$ have good accuracy for small qu to N. On the other hand, the normal approximation of Z has good accuracy for large qu, and the first order asymptotic expansion has good accuracy in almost all range of (qu, N) except for small qu.

6 Conclusion

In this paper, we proposed the likelihood ratio criterion for testing the BCS structure hypothesis:

$$H_0 : \boldsymbol{\Sigma} = \boldsymbol{I}_u \otimes (\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1) + \boldsymbol{J}_u \otimes \boldsymbol{\Sigma}_1 \quad \text{vs.} \quad H_1 \ne H_0.$$

We gave the hth moment of the likelihood ratio statistic under null hypothesis. And we derived the coefficient of Bartlett correction and an asymptotic expansion of the null distribution under the large sample framework A1 : q, u are fixed, $N \to \infty$. Further high-dimensional approximations of the null distribution under a high-dimensional framework:

$$\text{A2} : q, u, N \to \infty, \ \frac{qu}{N} \to c \in (0, 1)$$

were derived. Finally, we investigate the accuracies of asymptotic expansions proposed in this paper by the Monte Carlo simulation.

Appendix

A.1. MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Gamma}$

The likelihood function to obtain MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Gamma}$ has the following form:

$$\begin{aligned} L(\boldsymbol{\mu}, \boldsymbol{\Gamma}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) &= \prod_{i=1}^{N} f(\boldsymbol{x}_i) \\ &= (2\pi)^{-\frac{Nqu}{2}} |\boldsymbol{\Gamma}|^{-\frac{N}{2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{N}(\boldsymbol{x}_i - \boldsymbol{\mu})'\boldsymbol{\Gamma}^{-1}(\boldsymbol{x}_i - \boldsymbol{\mu})\right] \\ &= (2\pi)^{-\frac{Nqu}{2}} |\boldsymbol{\Gamma}|^{-\frac{N}{2}} \exp\left[-\frac{1}{2}\left(\text{tr}(\boldsymbol{\Gamma}^{-1}\boldsymbol{V}) - N\text{tr}\{\boldsymbol{\Gamma}^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu})(\bar{\boldsymbol{x}} - \boldsymbol{\mu})'\}\right)\right], \end{aligned}$$

where $\bar{\boldsymbol{x}} = N^{-1}\sum_{r=1}^{N} \boldsymbol{x}_r$ and $\boldsymbol{V} = \sum_{r=1}^{N}(\boldsymbol{x}_r - \bar{\boldsymbol{x}})(\boldsymbol{x}_r - \bar{\boldsymbol{x}})'$. Hence, MLE of $\boldsymbol{\mu}$ is given by $\widehat{\boldsymbol{\mu}} = \bar{\boldsymbol{x}}$. Let $\boldsymbol{C}_0 = \sum_{r=1}^{u} \boldsymbol{V}_{rr}$, $\boldsymbol{C}_1 = \sum\sum_{r \neq s} \boldsymbol{V}_{rs}$, $\boldsymbol{A} = \boldsymbol{\Delta}_1^{-1}$ and $\boldsymbol{B} = \frac{1}{u}\left[\boldsymbol{\Delta}_0^{-1} - \boldsymbol{\Delta}_1^{-1}\right]$. Then,

$$\begin{aligned} \text{tr}\left[\boldsymbol{\Gamma}^{-1}\boldsymbol{V}\right] &= \text{tr}\left[(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C}_0 + \boldsymbol{B}\boldsymbol{C}_1\right] \\ &= \text{tr}\left[\{\boldsymbol{I}_{Nu} \otimes \boldsymbol{A} + \boldsymbol{I}_N \otimes \boldsymbol{J}_u \otimes \boldsymbol{B}\}\right. \\ &\quad \left.\cdot\left\{\boldsymbol{I}_{Nu} \otimes \frac{1}{Nu}\left(\boldsymbol{C}_0 - \frac{1}{(u-1)}\boldsymbol{C}_1\right) + \boldsymbol{I}_N \otimes \boldsymbol{J}_u \otimes \frac{1}{Nu(u-1)}\boldsymbol{C}_1\right\}\right] \\ &= N\text{tr}\left[\boldsymbol{\Gamma}^{-1}\left\{\boldsymbol{I}_u \otimes \frac{1}{Nu}\left(\boldsymbol{C}_0 - \frac{1}{(u-1)}\boldsymbol{C}_1\right) + \boldsymbol{J}_u \otimes \frac{1}{Nu(u-1)}\boldsymbol{C}_1\right\}\right]. \end{aligned}$$

Therefore, logarithm of $L(\widehat{\boldsymbol{\mu}}, \boldsymbol{\Gamma}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ can be expressed as:

$$\log L(\boldsymbol{\mu}, \boldsymbol{\Gamma}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = -\frac{Nqu}{2}\log(2\pi) - \frac{N}{2}\log|\boldsymbol{\Gamma}| - \frac{1}{2}\left(\text{tr}\left[\boldsymbol{\Gamma}^{-1}\boldsymbol{V}\right]\right).$$

Let $\boldsymbol{C} = \boldsymbol{I}_u \otimes \frac{1}{Nu}\left(\boldsymbol{C}_0 - \frac{1}{(u-1)}\boldsymbol{C}_1\right) + \boldsymbol{J}_u \otimes \frac{1}{Nu(u-1)}\boldsymbol{C}_1$. Using Lemma 3.2.2 of Anderson (1984), this maximum of $\boldsymbol{\Gamma}$ is reached when $\widehat{\boldsymbol{\Gamma}} = \boldsymbol{C}$.

A.2. Distributions of $\widehat{\boldsymbol{\Delta}}_0$ and $\widehat{\boldsymbol{\Delta}}_1$

Let $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)'$. Then, we can express

$$\begin{aligned} \bar{\boldsymbol{x}} &= \frac{1}{N}\sum_{r=1}^{N}\boldsymbol{x}_r = \frac{1}{N}\boldsymbol{X}'\boldsymbol{1}_N, \\ \boldsymbol{S} &= \frac{1}{N}\sum_{r=1}^{N}(\boldsymbol{x}_r - \bar{\boldsymbol{x}})(\boldsymbol{x}_r - \bar{\boldsymbol{x}})' = \frac{1}{N}\boldsymbol{X}'\boldsymbol{Q}\boldsymbol{X}, \end{aligned}$$

where $\boldsymbol{Q} = \boldsymbol{I}_N - \frac{1}{N}\boldsymbol{J}_N$. Moreover, let $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_u)$. Then, $\boldsymbol{S}$ can be written as

$$\begin{aligned}\boldsymbol{S} &= \frac{1}{N}\boldsymbol{X}'\boldsymbol{Q}\boldsymbol{X} \\ &= \frac{1}{N}\begin{pmatrix} \boldsymbol{X}_1'\boldsymbol{Q}_N\boldsymbol{X}_1 & \cdots & \boldsymbol{X}_1'\boldsymbol{Q}_N\boldsymbol{X}_u \\ \vdots & \ddots & \vdots \\ \boldsymbol{X}_u'\boldsymbol{Q}_N\boldsymbol{X}_1 & \cdots & \boldsymbol{X}_u'\boldsymbol{Q}_N\boldsymbol{X}_u \end{pmatrix} = \begin{pmatrix} \boldsymbol{S}_{11} & \cdots & \boldsymbol{S}_{1u} \\ \vdots & \ddots & \vdots \\ \boldsymbol{S}_{u1} & \cdots & \boldsymbol{S}_{uu} \end{pmatrix}.\end{aligned}$$

Let block diagonal matrix of $\boldsymbol{S}$ be denoted by

$$\boldsymbol{D}[\boldsymbol{S}] = \begin{pmatrix} \boldsymbol{S}_{11} & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{S}_{uu} \end{pmatrix} = \frac{1}{N}\begin{pmatrix} \boldsymbol{V}_{11} & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{V}_{uu} \end{pmatrix}.$$

Then, we rewrite $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$ as follows:

$$\begin{aligned}\boldsymbol{C}_0 &= N(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\boldsymbol{D}[\boldsymbol{S}](\boldsymbol{1}_u \otimes \boldsymbol{I}_q), \\ \boldsymbol{C}_1 &= N(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\{\boldsymbol{S} - \boldsymbol{D}[\boldsymbol{S}]\}(\boldsymbol{1}_u \otimes \boldsymbol{I}_q).\end{aligned}$$

Hence, we can obtain

$$\begin{aligned}\widehat{\boldsymbol{\Delta}}_0 &= \frac{1}{u}(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{1}_u \otimes \boldsymbol{I}_q), \\ \widehat{\boldsymbol{\Delta}}_1 &= \frac{1}{u-1}(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\left\{\boldsymbol{D}[\boldsymbol{S}] - \frac{1}{u}\boldsymbol{S}\right\}(\boldsymbol{1}_u \otimes \boldsymbol{I}_q).\end{aligned}$$

Moreover, let $\boldsymbol{B}_p = \frac{1}{p}\boldsymbol{J}_p$, we can express $\widehat{\boldsymbol{\Delta}}_0$ and $\widehat{\boldsymbol{\Delta}}_1$ as follows:

$$\widehat{\boldsymbol{\Delta}}_0 = (\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\boldsymbol{D}\left[(\boldsymbol{B}_u \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{B}_u \otimes \boldsymbol{I}_q)\right](\boldsymbol{1}_u \otimes \boldsymbol{I}_q), \tag{6.1}$$

$$\widehat{\boldsymbol{\Delta}}_1 = \frac{1}{u-1}(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\boldsymbol{D}\left[(\boldsymbol{Q}_u \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{Q}_u \otimes \boldsymbol{I}_q)\right](\boldsymbol{1}_u \otimes \boldsymbol{I}_q). \tag{6.2}$$

Proof. By using

$$(\boldsymbol{B}_u \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{B}_u \otimes \boldsymbol{I}_q) = \begin{pmatrix} \frac{1}{u^2}(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{1}_u \otimes \boldsymbol{I}_q) & \cdots & \frac{1}{u^2}(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{1}_u \otimes \boldsymbol{I}_q) \\ \vdots & \ddots & \vdots \\ \frac{1}{u^2}(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{1}_u \otimes \boldsymbol{I}_q) & \cdots & \frac{1}{u^2}(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{1}_u \otimes \boldsymbol{I}_q) \end{pmatrix},$$

we obtain

$$(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\boldsymbol{D}\left[(\boldsymbol{B}_u \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{B}_u \otimes \boldsymbol{I}_q)\right](\boldsymbol{1}_u \otimes \boldsymbol{I}_q) = \frac{1}{u}(\boldsymbol{1}_u' \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{1}_u \otimes \boldsymbol{I}_q).$$

Similary, note that

$$(\boldsymbol{Q}_u \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{Q}_u \otimes \boldsymbol{I}_q) = \boldsymbol{S} - (\boldsymbol{B}_u \otimes \boldsymbol{I}_q)\boldsymbol{S} - \boldsymbol{S}(\boldsymbol{B}_u \otimes \boldsymbol{I}_q) + (\boldsymbol{B}_u \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{B}_u \otimes \boldsymbol{I}_q),$$

we can derive

$$
\begin{aligned}
&(\mathbf{1}'_u \otimes \boldsymbol{I}_q)\boldsymbol{D}\left[(\boldsymbol{Q}_u \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{Q}_u \otimes \boldsymbol{I}_q)\right](\mathbf{1}_u \otimes \boldsymbol{I}_q) \\
&\quad = \sum_{r=1}^{u} \boldsymbol{S}_{rr} - \frac{1}{u}\sum_{r=1}^{u}\sum_{s=1}^{u} \boldsymbol{S}_{rs} \\
&\quad = (\mathbf{1}'_u \otimes \boldsymbol{I}_q)\boldsymbol{D}\left[\boldsymbol{S}\right](\mathbf{1}_u \otimes \boldsymbol{I}_q) - \frac{1}{u}(\mathbf{1}'_u \otimes \boldsymbol{I}_q)\boldsymbol{S}(\mathbf{1}_u \otimes \boldsymbol{I}_q) \\
&\quad = (\mathbf{1}'_u \otimes \boldsymbol{I}_q)\left\{\boldsymbol{D}[\boldsymbol{S}] - \frac{1}{u}\boldsymbol{S}\right\}(\mathbf{1}_u \otimes \boldsymbol{I}_q).
\end{aligned}
$$

□

Let $\boldsymbol{l}_q \in \mathbb{R}^q - \{\mathbf{0}\}$. Then, we can obtain

$$
\begin{aligned}
\boldsymbol{l}'_q(\mathbf{1}'_u \otimes \boldsymbol{I}_q)\boldsymbol{D}\left[(\boldsymbol{B}_u \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{B}_u \otimes \boldsymbol{I}_q)\right](\mathbf{1}_u \otimes \boldsymbol{I}_q)\boldsymbol{l}_q &= \frac{u}{N}\left(\frac{1}{u^2}\sum_{r=1}^{u}\sum_{s=1}^{u} \boldsymbol{l}'_q\boldsymbol{X}'_r\boldsymbol{Q}_N\boldsymbol{X}_s\boldsymbol{l}_q\right) \\
&= \operatorname{tr}\left[(\boldsymbol{B}_u \otimes \boldsymbol{l}'_q)\boldsymbol{S}(\boldsymbol{B}_u \otimes \boldsymbol{l}_q)\right], \qquad (6.3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\boldsymbol{l}'_q(\mathbf{1}'_u \otimes \boldsymbol{I}_q)\boldsymbol{D}\left[(\boldsymbol{Q}_u \otimes \boldsymbol{I}_q)\boldsymbol{S}(\boldsymbol{Q}_u \otimes \boldsymbol{I}_q)\right](\mathbf{1}_u \otimes \boldsymbol{I}_q)\boldsymbol{l}_q \\
&= \operatorname{tr}\left[(\boldsymbol{I}_u \otimes \boldsymbol{l}'_q)\boldsymbol{S}(\boldsymbol{I}_u \otimes \boldsymbol{l}_q)\right] - 2\operatorname{tr}\left[(\boldsymbol{B}_u \otimes \boldsymbol{l}'_q)\boldsymbol{S}(\boldsymbol{I}_u \otimes \boldsymbol{l}_q)\right] + \operatorname{tr}\left[(\boldsymbol{B}_u \otimes \boldsymbol{l}'_q)\boldsymbol{S}(\boldsymbol{B}_u \otimes \boldsymbol{l}_q)\right] \\
&= \operatorname{tr}\left[(\boldsymbol{Q}_u \otimes \boldsymbol{l}'_q)\boldsymbol{S}(\boldsymbol{Q}_u \otimes \boldsymbol{l}_q)\right], \qquad (6.4)
\end{aligned}
$$

and

$$
\begin{aligned}
(\boldsymbol{B}_u \otimes \boldsymbol{l}'_q)\boldsymbol{\Gamma}(\boldsymbol{B}_u \otimes \boldsymbol{l}_q) &= (\boldsymbol{B}_u \otimes \boldsymbol{l}'_q)\{\boldsymbol{I}_u \otimes (\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1) + \boldsymbol{J}_u \otimes \boldsymbol{\Sigma}_1\}(\boldsymbol{B}_u \otimes \boldsymbol{l}_q) \\
&= \frac{1}{u}\{\boldsymbol{l}'_q(\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1)\boldsymbol{l}_q\}\boldsymbol{J}_u + (\boldsymbol{l}'_q\boldsymbol{\Sigma}_1\boldsymbol{l}_q)\boldsymbol{J}_u \\
&= \left(\boldsymbol{l}'_q\boldsymbol{\Delta}_0\boldsymbol{l}_q\right)\boldsymbol{B}_u, \qquad (6.5) \\
(\boldsymbol{Q}_u \otimes \boldsymbol{l}'_q)\boldsymbol{\Gamma}(\boldsymbol{Q}_u \otimes \boldsymbol{l}_q) &= (\boldsymbol{Q}_u \otimes \boldsymbol{l}'_q)\{\boldsymbol{I}_u \otimes (\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1) + \boldsymbol{J}_u \otimes \boldsymbol{\Sigma}_1\}(\boldsymbol{Q}_u \otimes \boldsymbol{l}_q) \\
&= \{\boldsymbol{l}'_q(\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1)\boldsymbol{l}_q\}\boldsymbol{Q}_u + (\boldsymbol{l}'_q\boldsymbol{\Sigma}_1\boldsymbol{l}_q)\boldsymbol{Q}_u\boldsymbol{J}_u\boldsymbol{Q}_u \\
&= \left(\boldsymbol{l}'_q\boldsymbol{\Delta}_1\boldsymbol{l}_q\right)\boldsymbol{Q}_u, \qquad (6.6)
\end{aligned}
$$

respectively. To derive the distributions of $\widehat{\boldsymbol{\Delta}}_0$ and $\widehat{\boldsymbol{\Delta}}_1$, we need the following two lemmas which are referred to in Rao (1973, 8b.2(ii)) and Klein and Žežula (2010, Lemma 2).

Lemma 1. *Let $\boldsymbol{\Sigma}$ is a $p \times p$ matrix . Then, it is true that*

$$
\boldsymbol{W} \sim W_p(f, \boldsymbol{\Sigma}) \Leftrightarrow \boldsymbol{l}'_p\boldsymbol{W}\boldsymbol{l}_p/\boldsymbol{l}'_p\boldsymbol{\Sigma}\boldsymbol{l}_p \sim \chi^2_f
$$

for every $\boldsymbol{l}_p$ satisfying $\boldsymbol{l}'_p\boldsymbol{\Sigma}\boldsymbol{l}_p \neq 0$

Lemma 2. *Let $\boldsymbol{W} \sim W_p(f, \boldsymbol{\Sigma})$, $\boldsymbol{\Sigma}$ is a $p \times p$ positive definite and $\boldsymbol{T}$ is a $k \times p$ matrix. Also let $r(\boldsymbol{T}) = \operatorname{rank}(\boldsymbol{T})$ and $\ell_1, \ldots, \ell_{r(\boldsymbol{T})} > 0$ are the characteristic roots of $\boldsymbol{T}\boldsymbol{W}\boldsymbol{T}'$. Then,*

$$
\operatorname{tr}\left(\boldsymbol{T}\boldsymbol{W}\boldsymbol{T}'\right) \sim \sum_{i=1}^{r(\boldsymbol{T})} \ell_i\chi^2_f.
$$

From lemma 2, $N\boldsymbol{S} \sim W_{qu}(n, \boldsymbol{\Gamma})$,

$$N\mathrm{tr}\left[(\boldsymbol{Q}_u \otimes \boldsymbol{l}_q')\boldsymbol{S}(\boldsymbol{Q}_u \otimes \boldsymbol{l}_q)\right] \sim \boldsymbol{l}_q'\boldsymbol{\Delta}_1\boldsymbol{l}_q\chi^2_{n(u-1)} \iff \boldsymbol{l}_q'\left(N(u-1)\widehat{\boldsymbol{\Delta}}_1\right)\boldsymbol{l}_q/\boldsymbol{l}_q'\boldsymbol{\Delta}_1\boldsymbol{l}_q \sim \chi^2_{n(u-1)}$$
$$\iff N(u-1)\widehat{\boldsymbol{\Delta}}_1 \sim W_{qu}(n(u-1), \boldsymbol{\Delta}_1),$$

and, similarly, $N\widehat{\boldsymbol{\Delta}}_0 \sim W_q(n, \boldsymbol{\Delta}_0)$. Moreover, $\boldsymbol{B}_u\boldsymbol{Q}_u = \boldsymbol{0}$, $\widehat{\boldsymbol{\Delta}}_0$ and $\widehat{\boldsymbol{\Delta}}_1$ are mutually independent.

A.3. Order of cumulant
We show that the order of $\kappa^{(r)}$ in a similar way to Kato et al. (2010). To prove the order of $\kappa^{(r)}$, we use the following lemmas:

Lemma 3. *Let p and r be positive integer such that $r > 2$. Then, for a positive real number A such that $A - p > 1/2$,*

$$\sum_{j=1}^{p} \frac{j}{(A-j)^r} \le \frac{p}{r-1}\left\{\frac{1}{(A-p-1/2)^{r-1}} - \frac{1}{(A-1/2)^{r-1}}\right\},$$
$$\sum_{j=1}^{p} \frac{1}{(A-j)^r} \le \frac{1}{r-1}\left\{\frac{1}{(A-p-1/2)^{r-1}} - \frac{1}{(A-1/2)^{r-1}}\right\}.$$

Lemma 4. *For $A > 0$, $v \ge 0$, $a > 0$ and any positive integer r,*

$$\frac{1}{A^r} - \frac{1}{(A+v)^r} \le \frac{vr}{A^r(A+v)},$$
$$\frac{1}{A^r} - \frac{1}{(A+v)^r a} \le \frac{1}{(A+v)^r} + \frac{vr}{A^r(A+v)}.$$

For $r \ge 1$, we can express $\kappa^{(r+1)}$ as follows:

$$\begin{aligned}\kappa^{(r+1)} &= (-1)^{r+1}\left\{\sum_{j=q}^{qu-1}\psi^{(r)}\left(\frac{1}{2}(n-j)\right) - \sum_{j=0}^{q-1}\psi^{(r)}\left(\frac{1}{2}(n(u-1)-j)\right)(u-1)^{r+1}\right\}\\
&= (-1)^{r+1}\sum_{j=q}^{qu-1}\left\{\psi^{(r)}\left(\frac{1}{2}(n-j)\right) - \frac{(u-1)^r}{q}\sum_{s=0}^{q-1}\psi^{(r)}\left(\frac{1}{2}(n(u-1)-s)\right)\right\}\\
&= \frac{(-1)^{r+1}}{q}\sum_{s=0}^{q-1}\sum_{j=q}^{qu-1}\left\{\psi^{(r)}\left(\frac{1}{2}(n-j)\right) - (u-1)^r\psi^{(r)}\left(\frac{1}{2}(n(u-1)-s)\right)\right\}.\end{aligned}$$

By trigonometric inequality, it holds that

$$|\kappa^{(r+1)}| \le \frac{1}{q}\sum_{s=0}^{q-1}\sum_{j=q}^{qu-1}\left|\psi^{(r)}\left(\frac{1}{2}(n-j)\right) - (u-1)^r\psi^{(r)}\left(\frac{1}{2}(n(u-1)-s)\right)\right|.$$

From the Stirling formula, there exists $\theta \in (0,1)$ such that

$$\psi(z) = \log z - \frac{1}{2z} + \frac{\theta}{12z^2},$$

and so

$$\psi^{(r)}\left(\frac{1}{2}(n-j)\right) = (-1)^{r+1}\left\{\frac{2^r(r-1)!}{(n-j)^r} + \frac{2^r r!}{(n-j)^{r+1}} + \frac{2^r\theta_1(r+1)!}{3(n-j)^{r+2}}\right\},$$

$$\psi^{(r)}\left(\frac{1}{2}(n(u-1)-s)\right) = (-1)^{r+1}\left\{\frac{2^r(r-1)!}{(n(u-1)-s)^r} + \frac{2^r r!}{(n(u-1)-s)^{r+1}} + \frac{2^r\theta_2(r+1)!}{3(n(u-1)-s)^{r+2}}\right\},$$

for some θ_1, θ_2. Hence, we have

$$\left|\psi^{(r)}\left(\frac{1}{2}(n-j)\right) - (u-1)^r\psi^{(r)}\left(\frac{1}{2}(n(u-1)-s)\right)\right| \le B_1 + B_2 + B_3,$$

where

$$\begin{aligned}
B_1 &= 2^r(r-1)!\left|\frac{1}{(n-j)^r} - \frac{(u-1)^r}{(n(u-1)-s)^r}\right|,\\
B_2 &= 2^r r!\left|\frac{1}{(n-j)^{r+1}} - \frac{(u-1)^r}{(n(u-1)-s)^{r+1}}\right|,\\
B_3 &= \frac{2^r(r+1)!}{3}\left|\frac{\theta_1}{(n-j)^{r+2}} - \frac{\theta_2(u-1)^r}{(n(u-1)-s)^{r+2}}\right|.
\end{aligned}$$

For $u \ge 2$, $j(u-1)-s > 0$.Then,

$$\begin{aligned}
B_1 &= 2^r(r-1)!\left\{\frac{1}{(n-j)^r} - \frac{(u-1)^r}{(n(u-1)-s)^r}\right\},\\
B_2 &= 2^r r!\left\{\frac{1}{(n-j)^{r+1}} - \frac{(u-1)^r}{(n(u-1)-s)^{r+1}}\right\},\\
B_3 &< \frac{2^r(r+1)!}{3(n-j)^{r+2}}.
\end{aligned}$$

Applying Lemmas 3 and 4, we can obtain

$$\begin{aligned}
\sum_{j=q}^{qu-1} B_1 &\le 2^r(r-1)!\sum_{j=q}^{qu-1}\left\{\frac{1}{(n-j)^r} - \frac{1}{n^r}\right\}\\
&\le 2^r(r-1)!\sum_{j=q}^{qu-1}\frac{jr}{(n-j)^r n}\\
&= 2^r r!\frac{q-1}{n}\sum_{j=1}^{q(u-1)}\frac{1}{(n-q+1-j)^r} + 2^r r!\frac{1}{n}\sum_{j=1}^{q(u-1)}\frac{j}{(n-q+1-j)^r}\\
&\le 2^r r!\frac{q-1}{n}\frac{1}{r-1}\left\{\frac{1}{(n-q+1/2-q(u-1))^{r-1}} - \frac{1}{(n-q+1/2)^{r-1}}\right\}\\
&\quad + 2^r r!\frac{q(u-1)}{n}\frac{1}{r-1}\left\{\frac{1}{(n-q+1/2-q(u-1))^{r-1}} - \frac{1}{(n-q+1/2)^{r-1}}\right\}\\
&\le 2^r r!\frac{qu-1}{n}\frac{1}{r-1}\frac{q(u-1)(r-1)}{(n-q+1/2-q(u-1))^{r-1}(n-q+1/2)}\\
&= 2^r r!\frac{qu-1}{n}\frac{q(u-1)}{(n-qu+1/2)^{r-1}(n-q+1/2)} \qquad (6.7)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=q}^{qu-1} B_2 &= 2^r r! \sum_{j=q}^{qu-1} \left\{ \frac{1}{(n-j)^{r+1}} - \frac{1}{n^{r+1}(u-1)} \right\} \\
&\le 2^r r! \sum_{j=q}^{qu-1} \left\{ \frac{1}{n^{r+1}} + \frac{j(r+1)}{(n-j)^{r+1}n} \right\} \\
&= 2^r r! \frac{q(u-1)}{n^{r+1}} + 2^r (r+1)! \sum_{j=q}^{qu-1} \frac{j}{(n-j)^{r+1}n} \\
&= 2^r r! \frac{q(u-1)}{n^{r+1}} + 2^r (r+1)! \frac{q-1}{n} \sum_{j=1}^{q(u-1)} \frac{1}{(n-q+1-j)^{r+1}} \\
&\quad + 2^r (r+1)! \frac{1}{n} \sum_{j=1}^{q(u-1)} \frac{j}{(n-q+1-j)^{r+1}} \\
&\le 2^r r! \frac{q(u-1)}{n^{r+1}} + 2^r (r+1)! \frac{q-1}{n} \frac{1}{r} \left\{ \frac{1}{(n-q+1/2-q(u-1))^r} - \frac{1}{(n-q+1/2)^r} \right\} \\
&\quad + 2^r (r+1)! \frac{q(u-1)}{n} \frac{1}{r} \left\{ \frac{1}{(n-q+1/2-q(u-1))^r} - \frac{1}{(n-q+1/2)^r} \right\} \\
&\le 2^r r! \frac{q(u-1)}{n^{r+1}} + 2^r (r+1)! \frac{qu-1}{n} \frac{1}{r} \frac{q(u-1)r}{(n-q+1/2-q(u-1))^r (n-q+1/2)} \\
&= 2^r r! \frac{q(u-1)}{n^{r+1}} + 2^r (r+1)! \frac{qu-1}{n} \frac{q(u-1)}{(n-qu+1/2)^r (n-q+1/2)} \qquad (6.8)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=q}^{qu-1} B_3 &= \frac{2^r (r+1)!}{3} \sum_{j=q}^{qu-1} \frac{1}{(n-j)^{r+2}} \\
&= \frac{2^r (r+1)!}{3} \sum_{j=1}^{q(u-1)} \frac{1}{(n-q+1-j)^{r+2}} \\
&\le \frac{2^r (r+1)!}{3} \frac{1}{r+1} \left\{ \frac{1}{(n-q+1/2-q(u-1))^{r+1}} - \frac{1}{(n-q+1/2)^{r+1}} \right\} \\
&\le \frac{2^r (r+1)!}{3} \frac{q(u-1)(r+1)}{(n-q+1/2-q(u-1))^{r+1}(n-q+1/2)} \\
&= \frac{2^r (r+1)!}{3} \frac{q(u-1)}{(n-qu+1/2)^{r+1}(n-q+1/2)}. \qquad (6.9)
\end{aligned}$$

From inequalities (6.7), (6.8) and (6.9), we can prove $|\kappa^{(r+1)}| = \mathrm{O}^*_{r-1}$.

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Table 1: Actual probabilities of χ^2 approximation and the asymptotic expansion of $-2\rho \log \lambda$ for $N = 20$

qu	q	u	0.1	0.05	0.50	0.95	0.99
4	2	2	0.0101 (0.0100)	0.0502 (0.0501)	0.5002 (0.4996)	0.9495 (0.9493)	0.9899 (0.9898)
6	3	2	0.0105 (0.0103)	0.0519 (0.0510)	0.5036 (0.4997)	0.9505 (0.4993)	0.9901 (0.9898)
	2	3	0.0108 (0.0104)	0.0531 (0.0514)	0.5063 (0.4996)	0.9512 (0.9492)	0.9904 (0.9898)
8	4	2	0.0114 (0.0106)	0.0546 (0.0516)	0.5108 (0.4985)	0.9515 (0.9479)	0.9902 (0.9892)
	2	4	0.0125 (0.0111)	0.0580 (0.0527)	0.5204 (0.4997)	0.9536 (0.9479)	0.9906 (0.9891)
10	5	2	0.0138 (0.0116)	0.0616 (0.0540)	0.5288 (0.5000)	0.9549 (0.9468)	0.9911 (0.9889)
	2	5	0.0161 (0.0124)	0.0696 (0.0564)	0.5499 (0.5045)	0.9590 (0.9470)	0.9920 (0.9888)
12	6	2	0.0184 (0.0132)	0.0758 (0.0580)	0.5596 (0.5015)	0.9604 (0.9448)	0.9923 (0.9881)
	2	6	0.0250 (0.0153)	0.0943 (0.0643)	0.6007 (0.5157)	0.9676 (0.9473)	0.9939 (0.9886)
	4	3	0.0219 (0.0145)	0.0859 (0.0618)	0.5840 (0.5107)	0.9646 (0.9464)	0.9932 (0.9883)
	3	4	0.0236 (0.0149)	0.0903 (0.0630)	0.5927 (0.5125)	0.9664 (0.9468)	0.9936 (0.9885)
14	7	2	0.0302 (0.0169)	0.1065 (0.0671)	0.6177 (0.5113)	0.9693 (0.9427)	0.9944 (0.9871)
	2	7	0.0469 (0.0220)	0.1466 (0.0812)	0.6838 (0.5445)	0.9789 (0.9501)	0.9964 (0.9889)
16	8	2	0.0678 (0.0283)	0.1854 (0.0924)	0.7194 (0.5453)	0.9821 (0.9451)	0.9969 (0.9868)
	2	8	0.1167 (0.0430)	0.2743 (0.1268)	0.8056 (0.6120)	0.9908 (0.9601)	0.9986 (0.9910)
	4	4	0.1042 (0.0394)	0.2537 (0.1194)	0.7892 (0.6009)	0.9895 (0.9580)	0.9983 (0.9905)
18	9	2	0.2283 (0.0845)	0.4159 (0.1908)	0.8714 (0.6554)	0.9947 (0.9599)	0.9992 (0.9901)
	2	9	0.3716 (0.1454)	0.5809 (0.2898)	0.9382 (0.7634)	0.9984 (0.9801)	0.9998 (0.9957)
	6	3	0.3051 (0.1184)	0.5086 (0.2486)	0.9132 (0.7248)	0.9972 (0.9739)	0.9996 (0.9941)
	3	6	0.3622 (0.1420)	0.5703 (0.2840)	0.9353 (0.7580)	0.9982 (0.9795)	0.9998 (0.9955)

Table 2: Actual probabilities of χ^2 approximation and the asymptotic expansion of $-2\rho\log\lambda$ for $N = 100$

qu	q	u	0.1	0.05	0.50	0.95	0.99
10	5	2	0.0099 (0.0099)	0.0501 (0.0499)	0.4992 (0.4986)	0.9499 (0.9497)	0.9901 (0.9901)
	2	5	0.0102 (0.0101)	0.0504 (0.0501)	0.5000 (0.4988)	0.9498 (0.9494)	0.9899 (0.9898)
20	10	2	0.0106 (0.0102)	0.0519 (0.0505)	0.5061 (0.5001)	0.9511 (0.9496)	0.9903 (0.9898)
	2	10	0.0110 (0.0103)	0.0535 (0.0508)	0.5109 (0.5002)	0.9521 (0.9493)	0.9906 (0.9898)
	5	4	0.0108 (0.0101)	0.0534 (0.0510)	0.5092 (0.4997)	0.9518 (0.9494)	0.9904 (0.9898)
	4	5	0.0108 (0.0101)	0.0531 (0.0505)	0.5102 (0.5004)	0.9520 (0.9494)	0.9905 (0.9898)
30	15	2	0.0119 (0.0104)	0.0572 (0.0511)	0.5229 (0.5001)	0.9550 (0.9492)	0.9912 (0.9896)
	2	15	0.0136 (0.0106)	0.0627 (0.0516)	0.5412 (0.5027)	0.9591 (0.9497)	0.9924 (0.9899)
	10	3	0.0128 (0.0105)	0.0596 (0.0513)	0.5316 (0.5012)	0.9569 (0.9494)	0.9918 (0.9898)
	3	10	0.0135 (0.0106)	0.0627 (0.0519)	0.5400 (0.5017)	0.9589 (0.9495)	0.9922 (0.9898)
	6	5	0.0131 (0.0105)	0.0611 (0.0513)	0.5373 (0.5019)	0.9586 (0.9498)	0.9920 (0.9898)
	5	6	0.0133 (0.0105)	0.0619 (0.0516)	0.5385 (0.5018)	0.9585 (0.9495)	0.9921 (0.9897)
40	20	2	0.0158 (0.0110)	0.0703 (0.0525)	0.5610 (0.5014)	0.9629 (0.9487)	0.9931 (0.9894)
	2	20	0.0215 (0.0117)	0.0878 (0.0554)	0.6060 (0.5095)	0.9709 (0.9502)	0.9950 (0.9898)
	10	4	0.0191 (0.0112)	0.0815 (0.0538)	0.5930 (0.5072)	0.9685 (0.9496)	0.9944 (0.9897)
	4	10	0.0207 (0.0115)	0.0865 (0.0547)	0.6037 (0.5087)	0.9706 (0.9498)	0.9948 (0.9898)
	8	5	0.0202 (0.0115)	0.0842 (0.0544)	0.5973 (0.5079)	0.9692 (0.9495)	0.9946 (0.9896)
	5	8	0.0209 (0.0116)	0.0866 (0.0550)	0.6037 (0.5094)	0.9705 (0.9499)	0.9948 (0.9898)
50	25	2	0.0262 (0.0119)	0.1020 (0.0551)	0.6364 (0.5096)	0.9754 (0.9482)	0.9958 (0.9892)
	2	25	0.8891 (0.8891)	0.8891 (0.8891)	0.8891 (0.8891)	0.9859 (0.9533)	0.9980 (0.9904)
	10	5	0.3100 (0.3100)	0.3100 (0.3100)	0.7060 (0.5262)	0.9844 (0.9530)	0.9977 (0.9903)
	5	10	0.7972 (0.7972)	0.7972 (0.7972)	0.7972 (0.7972)	0.9857 (0.9531)	0.9979 (0.9906)

Table 3: Actual probabilities of χ^2 approximation and the asymptotic expansion of $-2\rho \log \lambda$ for $N = 100$

qu	q	u	0.1	0.05	0.50	0.95	0.99
60	30	2	0.0606 (0.0150)	0.1861 (0.0631)	0.7601 (0.5221)	0.9894 (0.9499)	0.9985 (0.9894)
	2	30	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	15	4	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	4	15	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	10	6	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	6	10	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
70	35	2	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	2	35	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	10	7	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	7	10	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
80	40	2	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	2	40	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	20	4	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	4	20	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	10	8	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	8	10	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
90	45	2	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	2	45	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	15	6	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	6	15	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	10	9	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
	9	10	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)

Table 4: Actual probabilities of asymptotic normality and the first order asymptotic expansion of Z for $N = 20$

qu	q	u	0.1	0.05	0.50	0.95	0.99
4	2	2	0.0318 (0.0110)	0.0702 (0.0559)	0.4057 (0.4964)	1.0000 (0.9252)	1.0000 (0.9949)
6	3	2	0.0263 (0.0105)	0.0678 (0.0529)	0.4377 (0.4985)	0.9914 (0.9423)	1.0000 (0.9898)
	2	3	0.0233 (0.0103)	0.0656 (0.0517)	0.4511 (0.4993)	0.9803 (0.9458)	0.9999 (0.9896)
8	4	2	0.0228 (0.0104)	0.0647 (0.0522)	0.4522 (0.4999)	0.9789 (0.9465)	0.9999 (0.9900)
	2	4	0.0199 (0.0101)	0.0617 (0.0515)	0.4643 (0.5003)	0.9697 (0.9483)	0.9987 (0.9901)
10	5	2	0.0210 (0.0101)	0.0628 (0.0514)	0.4618 (0.4987)	0.9725 (0.9477)	0.9992 (0.9899)
	2	5	0.0180 (0.0102)	0.0600 (0.0508)	0.4721 (0.4997)	0.9651 (0.9490)	0.9974 (0.9900)
12	6	2	0.0195 (0.0101)	0.0615 (0.0511)	0.4665 (0.4999)	0.9685 (0.9485)	0.9984 (0.9901)
	2	6	0.0170 (0.0102)	0.0591 (0.0504)	0.4756 (0.5001)	0.9626 (0.9491)	0.9964 (0.9900)
	4	3	0.0178 (0.0100)	0.0598 (0.0506)	0.4738 (0.4995)	0.9640 (0.9491)	0.9970 (0.9902)
	3	4	0.0174 (0.0101)	0.0591 (0.0507)	0.4738 (0.5009)	0.9633 (0.9492)	0.9967 (0.9900)
14	7	2	0.0186 (0.0102)	0.0604 (0.0510)	0.4694 (0.5004)	0.9661 (0.9492)	0.9978 (0.9901)
	2	7	0.0165 (0.0101)	0.0582 (0.0506)	0.4774 (0.5002)	0.9612 (0.9495)	0.9960 (0.9901)
16	8	2	0.0187 (0.0100)	0.0605 (0.0507)	0.4707 (0.4996)	0.9655 (0.9491)	0.9974 (0.9904)
	2	8	0.0165 (0.0099)	0.0579 (0.0509)	0.4776 (0.4998)	0.9614 (0.9492)	0.9958 (0.9903)
	4	4	0.0168 (0.0101)	0.0582 (0.0509)	0.4777 (0.4998)	0.9622 (0.9489)	0.9961 (0.9902)
18	9	2	0.0204 (0.0098)	0.0611 (0.0519)	0.4640 (0.4993)	0.9685 (0.9494)	0.9982 (0.9916)
	2	9	0.0186 (0.0097)	0.0593 (0.0516)	0.4710 (0.4990)	0.9646 (0.9498)	0.9968 (0.9913)
	6	3	0.0191 (0.0097)	0.0599 (0.0516)	0.4693 (0.4989)	0.9658 (0.9498)	0.9973 (0.9915)
	3	6	0.0187 (0.0097)	0.0596 (0.0513)	0.4733 (0.4972)	0.9650 (0.9491)	0.9969 (0.9912)

Table 5: Actual probabilities of asymptotic normality and the first order asymptotic expansion of Z for $N = 100$

qu	q	u	0.1	0.05	0.50	0.95	0.99
10	5	2	0.0204 (0.0103)	0.0624 (0.0514)	0.4612 (0.5012)	0.9716 (0.9481)	0.9991 (0.9897)
	2	5	0.0178 (0.0101)	0.0597 (0.0507)	0.4721 (0.5011)	0.9639 (0.9494)	0.9972 (0.9900)
20	10	2	0.0156 (0.0100)	0.0573 (0.0502)	0.4814 (0.4998)	0.9594 (0.9497)	0.9954 (0.9900)
	2	10	0.0140 (0.0100)	0.0555 (0.0499)	0.4867 (0.5001)	0.9562 (0.9501)	0.9938 (0.9900)
	5	4	0.0141 (0.0101)	0.0560 (0.0498)	0.4855 (0.5005)	0.9565 (0.9500)	0.9940 (0.9901)
	4	5	0.0140 (0.0102)	0.0554 (0.0502)	0.4864 (0.4998)	0.9564 (0.9500)	0.9938 (0.9900)
30	15	2	0.0137 (0.0100)	0.0553 (0.0500)	0.4874 (0.5000)	0.9563 (0.9498)	0.9936 (0.9901)
	2	15	0.0127 (0.0100)	0.0534 (0.0502)	0.4917 (0.4995)	0.9544 (0.9497)	0.9928 (0.9898)
	10	3	0.0131 (0.0100)	0.0539 (0.0503)	0.4899 (0.5001)	0.9548 (0.9498)	0.9930 (0.9899)
	3	10	0.0126 (0.0100)	0.0538 (0.0500)	0.4906 (0.5004)	0.9543 (0.9498)	0.9926 (0.9899)
	6	5	0.0126 (0.0101)	0.0534 (0.0505)	0.4908 (0.4999)	0.9548 (0.9495)	0.9927 (0.9900)
	5	6	0.0126 (0.0101)	0.0537 (0.0503)	0.4907 (0.5001)	0.9544 (0.9499)	0.9927 (0.9900)
40	20	2	0.0129 (0.0099)	0.0541 (0.0499)	0.4905 (0.4998)	0.9546 (0.9497)	0.9928 (0.9900)
	2	20	0.0121 (0.0099)	0.0532 (0.0498)	0.4934 (0.4999)	0.9531 (0.9499)	0.9920 (0.9900)
	10	4	0.0119 (0.0102)	0.0526 (0.0505)	0.4928 (0.4999)	0.9531 (0.9501)	0.9921 (0.9900)
	4	10	0.0119 (0.0101)	0.0527 (0.0501)	0.4928 (0.5003)	0.9530 (0.9502)	0.9920 (0.9900)
	8	5	0.0121 (0.0100)	0.0529 (0.0501)	0.4929 (0.5002)	0.9529 (0.9504)	0.9920 (0.9902)
	5	8	0.0121 (0.0100)	0.0531 (0.0499)	0.4937 (0.4995)	0.9531 (0.9501)	0.9919 (0.9900)
50	25	2	0.0124 (0.0099)	0.0531 (0.0502)	0.4932 (0.4990)	0.9537 (0.9499)	0.9922 (0.9900)
	2	25	0.0116 (0.0101)	0.0519 (0.0504)	0.4939 (0.5006)	0.9523 (0.9502)	0.9915 (0.9901)
	10	5	0.0115 (0.0102)	0.0523 (0.0502)	0.4951 (0.4992)	0.9529 (0.9497)	0.9917 (0.9900)
	5	10	0.0118 (0.0098)	0.0527 (0.0497)	0.4950 (0.4994)	0.9524 (0.9502)	0.9917 (0.9899)

Table 6: Actual probabilities of asymptotic normality and the first order asymptotic expansion of Z for $N = 100$

qu	q	u	0.1	0.05	0.50	0.95	0.99
60	30	2	0.0120 (0.0100)	0.0532 (0.0496)	0.4941 (0.4993)	0.9530 (0.9500)	0.9919 (0.9900)
	2	30	0.0114 (0.0100)	0.0518 (0.0503)	0.4955 (0.4998)	0.9519 (0.9501)	0.9913 (0.9901)
	15	4	0.0116 (0.0098)	0.0521 (0.0501)	0.4950 (0.4999)	0.9521 (0.9501)	0.9914 (0.9901)
	4	15	0.0114 (0.0100)	0.0519 (0.0501)	0.4952 (0.4999)	0.9523 (0.9498)	0.9916 (0.9899)
	10	6	0.0114 (0.0100)	0.0520 (0.0502)	0.4948 (0.5003)	0.9523 (0.9499)	0.9914 (0.9900)
	6	10	0.0112 (0.0103)	0.0515 (0.0505)	0.4957 (0.4995)	0.9523 (0.9499)	0.9914 (0.9900)
70	35	2	0.0120 (0.0098)	0.0530 (0.0496)	0.4948 (0.4993)	0.9527 (0.9500)	0.9918 (0.9900)
	2	35	0.0112 (0.0100)	0.0522 (0.0497)	0.4963 (0.4994)	0.9522 (0.9497)	0.9913 (0.9900)
	10	7	0.0114 (0.0099)	0.0519 (0.0499)	0.4952 (0.5005)	0.9521 (0.9498)	0.9913 (0.9900)
	7	10	0.0114 (0.0099)	0.0520 (0.0499)	0.4966 (0.4991)	0.9523 (0.9497)	0.9914 (0.9899)
80	40	2	0.0117 (0.0100)	0.0528 (0.0497)	0.4950 (0.4995)	0.9526 (0.9500)	0.9916 (0.9901)
	2	40	0.0114 (0.0099)	0.0519 (0.0500)	0.4957 (0.5001)	0.9520 (0.9498)	0.9913 (0.9899)
	20	4	0.0112 (0.0101)	0.0523 (0.0496)	0.4960 (0.4996)	0.9526 (0.9494)	0.9913 (0.9900)
	4	20	0.0113 (0.0100)	0.0515 (0.0503)	0.4950 (0.5008)	0.9516 (0.9502)	0.9912 (0.9900)
	10	8	0.0115 (0.0098)	0.0520 (0.0498)	0.4954 (0.5003)	0.9521 (0.9498)	0.9912 (0.9900)
	8	10	0.0113 (0.0099)	0.0516 (0.0503)	0.4955 (0.5004)	0.9516 (0.9503)	0.9912 (0.9901)
90	45	2	0.0119 (0.0100)	0.0525 (0.050)	0.4935 (0.5004)	0.9530 (0.9498)	0.9919 (0.9899)
	2	45	0.0114 (0.0099)	0.0522 (0.0499)	0.4954 (0.4997)	0.9522 (0.9500)	0.9914 (0.9900)
	15	6	0.0114 (0.0100)	0.0520 (0.0501)	0.4953 (0.4998)	0.9521 (0.9501)	0.9915 (0.9899)
	6	15	0.0115 (0.0100)	0.0521 (0.0501)	0.4956 (0.4995)	0.9521 (0.9502)	0.9915 (0.9900)
	10	9	0.0114 (0.0101)	0.0519 (0.0501)	0.4958 (0.4993)	0.9521 (0.9501)	0.9915 (0.9899)
	9	10	0.0115 (0.0100)	0.0520 (0.0500)	0.4955 (0.4997)	0.9521 (0.9502)	0.9915 (0.9899)