Modified Likelihood Ratio Tests in a One-way MANOVA with Monotone Missing Data

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Abstract

In this study, testing the equality of mean vectors in a one-way multivariate analysis of variance (MANOVA) is considered when each data set has a monotone pattern of missing observations. The likelihood ratio test (LRT) statistic in a one-way MANOVA with monotone missing data is given. Furthermore, modified LRT (MLRT) statistics with monotone missing data are proposed using the decomposition of the likelihood ratio (LR) and an asymptotic expansion for each decomposed LR. Finally, the accuracy and asymptotic behavior of the approximation for the chi-square distribution are investigated using a Monte Carlo simulation.

Key Words and Phrases: Asymptotic expansion, Chi-square distribution, Decomposition of likelihood ratio, Maximum likelihood estimator, Monte Carlo simulation.

1 Introduction

In this study, we deal with the LRT statistic to test the equality of mean vectors in a one-way MANOVA when each data set has a monotone pattern of missing observations. The LRTs for mean vectors with monotone missing data have been discussed by many authors. For the one-sample problem, Krishnamoorthy and Pannala (1998) gave the LR decomposition and provided comparisons with several approximation procedures. Then, Seko, Yamazaki and Seo (2012) gave the LRT statistic and the linear interpolation approximation to the null distribution in the two-step monotone missing case. For a discussion on developing estimation and testing procedures for the mean vector and the scale matrix of the elliptical distributions with monotone missing data, see Batsidis and Zografos (2006). MLRT statistics of the one-sample test for a normal mean vector with monotone missing data are obtained by Yagi, Seo and Srivastava (2016). As an extension of the result in Yagi, Seo and Srivastava (2016) to the two-sample problem, the LRT and the MLRT statistics are derived by Yagi, Hanusz and Seo (2016). Seko (2012) gave the LRT statistic for testing the equality of mean vectors with two-step monotone missing data, and discussed the asymptotic null distribution. In this study, we consider the LRT and the MLRT statistics in the one-way MANOVA with monotone missing data. In the case of the one-way MANOVA with non-missing data, it is well known that Wilks' Λ statistic is the LRT statistic, and its MLRT statistic is given (see, e.g., Srivastava (2002) and Fujikoshi, Ulyanov and Shimizu (2010)).

First, we give the LRT statistic for general monotone missing data and derive an asymptotic expansion for the upper percentile of the null distribution of the LRT statistic for a subvector, using a perturbation procedure. Second, using these results, we propose an MLRT statistic with monotone missing data. In order to establish the purpose of the study, we decompose the LRT statistic and derive an asymptotic expansion of the characteristic function of each decomposed LRT statistic for non-missing data sets. A related discussion of a test for a subvector and a decomposition with complete data was given by Siotani, Hayakawa and Fujikoshi (1985). For the non-normal case, Gupta, Xu and Fujikoshi (2006) derived the asymptotic expansion of the distribution of Rao's U-statistic.

The remainder of this paper is organized as follows. In Section 2, we give the LR for a one-way MANOVA, using the MLEs of the covariance matrix. Furthermore, we give the condition that the null distribution of the LRT statistic is asymptotically distributed as a chi-square distribution. In Section 3, in order to derive the MLRT statistics, we consider the decomposition of the LR as the products of independent LRs for a one-way MANOVA of the reduced dimension and those of the remaining subvectors with complete data. In Section 4, simulation results are presented to investigate the accuracy of the approximation to the chi-square distribution for the null distribution of the MLRT statistics. Finally, we state our conclusions.

2 LRT statistic

Let $\boldsymbol{x}_i^{(\ell)}$, $\ell = 1, 2, \ldots, m$ be a $p_i \times 1$ normal random vector with the mean vector $\boldsymbol{\mu}_i^{(\ell)}$ and the covariance matrix $\boldsymbol{\Sigma}_i (> O)$, where $\boldsymbol{\mu}_i^{(\ell)} = (\boldsymbol{\mu}^{(\ell)})_i = (\mu_1^{(\ell)}, \mu_2^{(\ell)}, \ldots, \mu_{p_i}^{(\ell)})'$, and $\boldsymbol{\Sigma}_i$ is the $p_i \times p_i$ principal submatrix of $\boldsymbol{\Sigma}(= \boldsymbol{\Sigma}_1)$, with $p = p_1 > p_2 > \cdots > p_k > 0$. Furthermore, assume that $\boldsymbol{x}_i^{(\ell)}$, $i = 1, 2, \ldots, k$ are mutually independent. Suppose that $\boldsymbol{x}_{i1}^{(\ell)}, \boldsymbol{x}_{i2}^{(\ell)}, \ldots, \boldsymbol{x}_{in_i^{(\ell)}}^{(\ell)}$ are independent and identically distributed samples from $\boldsymbol{x}_i^{(\ell)}$, $i = 1, 2, \ldots, k$, where $\nu_1 = \sum_{\ell=1}^m n_1^{(\ell)}$ and $\nu_1 - m > p$. Note that k denotes the number of steps. Then, the above data is referred to as k-step monotone missing data. For this notation and assumption, see Jinadasa and Tracy (1992) and Yagi and Seo (2016), among others. With regard to the partitions of $\boldsymbol{\Sigma}$, for $1 \leq i < j \leq k$, let $(\boldsymbol{\Sigma}_i)_j$ be the principal submatrix of $\boldsymbol{\Sigma}_i$ of order $p_j \times p_j$; then, we define

$$\boldsymbol{\Sigma}_i = (\boldsymbol{\Sigma}_1)_i, \quad \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_i & \boldsymbol{\Sigma}_{i2} \\ \boldsymbol{\Sigma}_{i2}' & \boldsymbol{\Sigma}_{i3} \end{pmatrix}, \quad \boldsymbol{\Sigma}_{i-1} = \begin{pmatrix} \boldsymbol{\Sigma}_i & \boldsymbol{\Sigma}_{(i-1,2)} \\ \boldsymbol{\Sigma}_{(i-1,2)}' & \boldsymbol{\Sigma}_{(i-1,3)} \end{pmatrix},$$

and

$$\boldsymbol{\Sigma}_{(i-1,3)\cdot i} = \boldsymbol{\Sigma}_{(i-1,3)} - \boldsymbol{\Sigma}_{(i-1,2)}' \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\Sigma}_{(i-1,2)}, \quad i = 2, 3, \ldots, k.$$

With the above notation and assumptions, we consider the LRT statistic for testing the equality of mean vectors in a one-way MANOVA with k-step monotone missing data. Suppose that $\boldsymbol{x}_{ij}^{(\ell)}$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n_i^{(\ell)}$, $\ell = 1, 2, \ldots, m$ are independent and identically distributed as $N_{p_i}(\boldsymbol{\mu}_i^{(\ell)}, \boldsymbol{\Sigma}_i)$. Then, the LR for

$$H_0 : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} = \cdots = \boldsymbol{\mu}^{(m)} \text{ vs. } H_1 : \text{not } H_0$$

can be obtained as

$$\lambda = \prod_{i=1}^k \left(\frac{|\widehat{\boldsymbol{\Sigma}}_i|}{|\widetilde{\boldsymbol{\Sigma}}_i|} \right)^{\frac{\nu_i}{2}}, \tag{1}$$

where $\nu_i = \sum_{\ell=1}^m n_i^{(\ell)}$, $i = 1, 2, \ldots, k$, $\widehat{\boldsymbol{\Sigma}}_i$ is the MLE of $\boldsymbol{\Sigma}_i$ under H_1, and $\widetilde{\boldsymbol{\Sigma}}_i$ is the MLE of $\boldsymbol{\Sigma}_i$ under H_0. Note that $\widehat{\boldsymbol{\Sigma}}$ is obtained by Theorem 1 in Yagi and Seo (2016). That is, setting

$$\overline{\boldsymbol{x}}_i^{(\ell)} = \frac{1}{n_i^{(\ell)}} \sum_{j=1}^{n_i^{(\ell)}} \boldsymbol{x}_{ij}^{(\ell)}, \quad \boldsymbol{E}_i^{(\ell)} = \sum_{j=1}^{n_i^{(\ell)}} (\boldsymbol{x}_{ij}^{(\ell)} - \overline{\boldsymbol{x}}_i^{(\ell)})(\boldsymbol{x}_{ij}^{(\ell)} - \overline{\boldsymbol{x}}_i^{(\ell)})', \quad i = 1, 2, \ldots, k,$$

$$\boldsymbol{d}_1^{(\ell)} = \overline{\boldsymbol{x}}_1^{(\ell)}, \quad \boldsymbol{d}_i^{(\ell)} = \frac{n_i^{(\ell)}}{N_{i+1}^{(\ell)}} \left[\overline{\boldsymbol{x}}_i^{(\ell)} - \frac{1}{N_i^{(\ell)}} \sum_{j=1}^{i-1} n_j^{(\ell)} (\overline{\boldsymbol{x}}_j^{(\ell)})_i \right], \quad i = 2, 3, \ldots, k,$$

$$N_1^{(\ell)} = 0, \; N_{i+1}^{(\ell)} = N_i^{(\ell)} + n_i^{(\ell)} \; (= \sum_{j=1}^{i} n_j^{(\ell)}), \quad i = 1, 2, \ldots, k,$$

$$M_i = \sum_{\ell=1}^{m} N_i^{(\ell)}, \quad i = 1, 2, \ldots, k+1,$$

the MLE of $\boldsymbol{\Sigma}$ under H_1 is given by

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{M_2} \sum_{\ell=1}^{m} \boldsymbol{H}_1^{(\ell)} + \sum_{\ell=1}^{m} \sum_{i=2}^{k} \frac{1}{M_{i+1}} \boldsymbol{F}_i^{[\mathrm{pl}]} \left[\boldsymbol{H}_i^{(\ell)} - \frac{\nu_i}{M_i} \boldsymbol{L}_{i-1,1}^{(\ell)} \right] \boldsymbol{F}_i^{[\mathrm{pl}]'},$$

where

$$\boldsymbol{H}_1^{(\ell)} = \boldsymbol{E}_1^{(\ell)}, \; \boldsymbol{H}_i^{(\ell)} = \boldsymbol{E}_i^{(\ell)} + \frac{N_i^{(\ell)} N_{i+1}^{(\ell)}}{n_i^{(\ell)}} \boldsymbol{d}_i^{(\ell)} \boldsymbol{d}_i^{(\ell)'}, \; i = 2, 3, \ldots, k,$$

$$\boldsymbol{L}_1^{(\ell)} = \boldsymbol{H}_1^{(\ell)}, \; \boldsymbol{L}_i^{(\ell)} = (\boldsymbol{L}_{i-1}^{(\ell)})_i + \boldsymbol{H}_i^{(\ell)}, \; i = 2, 3, \ldots, k,$$

$$\boldsymbol{L}_{i1}^{(\ell)} = (\boldsymbol{L}_i^{(\ell)})_{i+1}, \; \boldsymbol{L}_i^{(\ell)} = \begin{pmatrix} \boldsymbol{L}_{i1}^{(\ell)} & \boldsymbol{L}_{i2}^{(\ell)} \\ \boldsymbol{L}_{i2}^{(\ell)'} & \boldsymbol{L}_{i3}^{(\ell)} \end{pmatrix}, \; i = 1, 2, \ldots, k-1$$

and

$$\boldsymbol{F}_1^{[\mathrm{pl}]} = \boldsymbol{G}_1, \quad \boldsymbol{F}_i^{[\mathrm{pl}]} = \boldsymbol{F}_{i-1}^{[\mathrm{pl}]} \boldsymbol{G}_i^{[\mathrm{pl}]}, \quad i = 2, 3, \ldots, k,$$

$$\boldsymbol{G}_1 = \boldsymbol{I}_{p_1}, \quad \boldsymbol{G}_{i+1}^{[\mathrm{pl}]} = \begin{pmatrix} \boldsymbol{I}_{p_{i+1}} \\ \left(\sum_{\ell=1}^{m} \boldsymbol{L}_{i2}^{(\ell)} \right)' \left(\sum_{\ell=1}^{m} \boldsymbol{L}_{i1}^{(\ell)} \right)^{-1} \end{pmatrix}, \quad i = 1, 2, \ldots, k-1.$$

On the other hand, we define

$$\overline{\boldsymbol{x}}_i = \frac{1}{\nu_i} \sum_{\ell=1}^{2} \sum_{j=1}^{n_i^{(\ell)}} \boldsymbol{x}_{ij}^{(\ell)}, \quad \boldsymbol{E}_i = \sum_{\ell=1}^{2} \sum_{j=1}^{n_i^{(\ell)}} (\boldsymbol{x}_{ij}^{(\ell)} - \overline{\boldsymbol{x}}_i)(\boldsymbol{x}_{ij}^{(\ell)} - \overline{\boldsymbol{x}}_i)', \quad i = 1, 2, \ldots, k,$$

$$\boldsymbol{d}_1 = \overline{\boldsymbol{x}}_1, \quad \boldsymbol{d}_i = \frac{\nu_i}{M_{i+1}} \left[\overline{\boldsymbol{x}}_i - \frac{1}{M_i} \sum_{j=1}^{i-1} \nu_j (\overline{\boldsymbol{x}}_j)_i \right], \quad i = 2, 3, \ldots, k.$$

Then, in the derivation analogous to the MLE under H_1, the MLE of $\boldsymbol{\Sigma}$ under H_0 can be obtained as

$$\widetilde{\boldsymbol{\Sigma}} = \frac{1}{M_2} \boldsymbol{H}_1 + \sum_{i=2}^{k} \frac{1}{M_{i+1}} \boldsymbol{F}_i \left[\boldsymbol{H}_i - \frac{\nu_i}{M_i} \boldsymbol{L}_{i-1,1} \right] \boldsymbol{F}_i',$$

where

$$
\begin{aligned}
&\boldsymbol{H}_1 = \boldsymbol{E}_1, \quad \boldsymbol{H}_i = \boldsymbol{E}_i + \frac{M_i M_{i+1}}{\nu_i}\boldsymbol{d}_i\boldsymbol{d}_i', \quad i = 2, 3, \ldots, k,\\
&\boldsymbol{L}_1 = \boldsymbol{H}_1, \ \boldsymbol{L}_i = (\boldsymbol{L}_{i-1})_i + \boldsymbol{H}_i, \ i = 2, 3, \ldots, k,\\
&\boldsymbol{L}_{i1} = (\boldsymbol{L}_i)_{i+1}, \ \boldsymbol{L}_i = \begin{pmatrix} \boldsymbol{L}_{i1} & \boldsymbol{L}_{i2} \\ \boldsymbol{L}_{i2}' & \boldsymbol{L}_{i3} \end{pmatrix}, \ i = 1, 2, \ldots, k-1,\\
&\boldsymbol{G}_1 = \boldsymbol{I}_{p_1}, \ \boldsymbol{G}_{i+1} = \begin{pmatrix} \boldsymbol{I}_{p_{i+1}} \\ \boldsymbol{L}_{i2}'\boldsymbol{L}_{i1}^{-1} \end{pmatrix}, \ i = 1, 2, \ldots, k-1,\\
&\boldsymbol{F}_1 = \boldsymbol{G}_1, \ \boldsymbol{F}_i = \boldsymbol{F}_{i-1}\boldsymbol{G}_i, \ i = 2, 3, \ldots, k.
\end{aligned}
$$

Note that the above result essentially coincides with the one-sample case of Jinadasa and Tracy (1992). Using the above results, we can obtain the LRT statistic $-2\log\lambda$, the null distribution of which is asymptotically distributed as a χ^2 distribution with $(m-1)p$ degrees of freedom when $n_1^{(\ell)} \to \infty$, $\ell = 1, 2, \ldots, m$, with $n_1^{(\ell)}/n_1^{(1)}$: const. $\to \omega^{(\ell)} \in (0, \infty)$, $\ell = 2, 3, \ldots, m$ and ν_i/ν_1: const. $\to \zeta_i \in [0, \infty)$, $i = 2, 3, \ldots, k$.

3 Modified LRT statistic and its null distribution

In this section, we derive an MLRT statistic using a decomposition of the LR and an asymptotic expansion procedure. Let

$$
\lambda_1 = \left(\frac{|\widehat{\boldsymbol{\Sigma}}_k|}{|\widetilde{\boldsymbol{\Sigma}}_k|}\right)^{\frac{M_{k+1}}{2}}, \quad \lambda_i = \left(\frac{|\widehat{\boldsymbol{\Sigma}}_{(k-i+1,3)\cdot k-i+2}|}{|\widetilde{\boldsymbol{\Sigma}}_{(k-i+1,3)\cdot k-i+2}|}\right)^{\frac{M_{k-i+2}}{2}}, \quad i = 2, 3, \ldots, k.
$$

Then, λ in (1) can be expressed as

$$
\lambda = \prod_{i=1}^{k}\lambda_i,
$$

where λ_i, $i = 1, 2, \ldots, k$ are mutually independent. Furthermore, let $Q_i^* = -2\rho_i \log\lambda_i$, where

$$
\rho_i = 1 - \frac{1}{2M_{k-i+2}}(p_{k-i+1} + p_{k-i+2} + m + 2), \quad i = 1, 2, \ldots, k, \quad p_{k+1} = 0. \tag{2}
$$

Then, we obtain

$$
\Pr(Q_i^* \le x) = G_{(m-1)(p_{k-i+1}-p_{k-i+2})}(x) + O(M_{k-i+2}^{-2}), \quad i = 1, 2, \ldots, k,
$$

where $G_f(x)$ is the distribution function of a χ^2-variate with f degrees of freedom. Therefore, we propose

$$Q^* = \sum_{i=1}^{k} Q_i^*$$

as an MLRT statistic. Since

$$E[\exp\{it(Q_j^*)\}] = (1-2it)^{-\frac{1}{2}(m-1)(p_{k-j+1}-p_{k-j+2})} + O(M_{k-j+2}^{-2}), \quad j = 1, 2, \ldots, k,$$

we have

$$E[\exp\{it(Q^*)\}] = (1-2it)^{-\frac{1}{2}(m-1)p} + O(M_2^{-2}).$$

Therefore, $\Pr(Q^* \le x) = G_{(m-1)p}(x) + O(M_2^{-2})$. Note that the MLRT statistic, Q^*, converges to the χ^2 distribution much faster than the LRT statistic does.

The derivation of the above result of the Bartlett correction factor ρ_i, $i = 1, 2, \ldots, k$ in (2) is as follows. For simplicity, we consider the case of $k = 2$. That is, we derive ρ_1 and ρ_2. Here, we consider the following hypotheses:

$$H_{01} : \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} = \cdots = \boldsymbol{\mu}_2^{(m)} \text{ vs. } H_{11} : \text{ not } H_{01},$$

and

$$\begin{aligned} &H_{02} : \boldsymbol{A}\boldsymbol{\mu}_1^{(1)} = \boldsymbol{A}\boldsymbol{\mu}_1^{(2)} = \cdots = \boldsymbol{A}\boldsymbol{\mu}_1^{(m)} \text{ given } \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} = \cdots = \boldsymbol{\mu}_2^{(m)} \\ &\text{vs. } H_{12} : \boldsymbol{A}\boldsymbol{\mu}_1^{(i)} \ne \boldsymbol{A}\boldsymbol{\mu}_1^{(j)} \text{ for some } i \ne j \text{ given } \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} = \cdots = \boldsymbol{\mu}_2^{(m)}, \end{aligned}$$

where

$$\boldsymbol{\mu}_1^{(\ell)} = \begin{pmatrix} \boldsymbol{\mu}_2^{(\ell)} \\ \boldsymbol{A}\boldsymbol{\mu}_1^{(\ell)} \end{pmatrix} \begin{matrix} \}p_2 \\ \}p_1 - p_2 \end{matrix},$$

and $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I}_{p_1-p_2} \end{pmatrix}$ is a $(p_1 - p_2) \times p_1$ matrix. Then, λ_1 and λ_2 are equal to the LRs for H_{01} and H_{02}, respectively (see Figure 1).

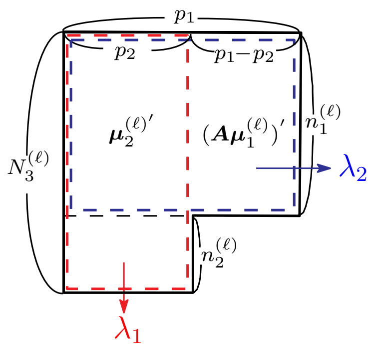

Figure 1. The LRs λ_1 and λ_2 in the case of two-step monotone missing data.

That is, $Q_1(= -2\log\lambda_1)$ and $Q_2(= -2\log\lambda_2)$ are the LRT statistics of the tests for a p_2-dimensional mean vector and a $(p_1 - p_2)$-dimensional subvector, respectively. The one-sample case is discussed by Krishnamoorthy and Pannala (1998).

We first consider the null distribution of $Q_1(= -2\log\lambda_1)$. Then, Q_1 is the LRT statistic for the test of the mean vector in a one-way MANOVA, where the data consist of complete data sets $((N_3^{(\ell)} \times p_2),\ \ell = 1, 2, \ldots, m)$. The Wilks' Λ statistic is given by

$$\Lambda = \frac{|\boldsymbol{S}_w|}{|\boldsymbol{S}_w + \boldsymbol{S}_b|},$$

where $\boldsymbol{S}_b$ and $\boldsymbol{S}_w$ are matrices of the sums of squares and the products (SSP matrices) from treatments (between groups) and errors (within groups), respectively (see Fujikoshi, Ulyanov and Shimizu (2010)). Therefore, the modified LRT statistic Q_1^* is given by

$$Q_1^* = -2\rho_1 \log \Lambda^{\frac{M_3}{2}},$$

where

$$\rho_1 = 1 - \frac{1}{2M_3}(p_2 + m + 2).$$

Furthermore, we have

$$\Pr(Q_1^* \le x) = G_{(m-1)p_2}(x) + O(M_3^{-2}).$$

Secondly, we derive an asymptotic expansion of the null distribution of $Q_2(= -2\log\lambda_2)$. For convenience, let $\boldsymbol{y}_1^{(\ell)},\ \boldsymbol{y}_2^{(\ell)}, \ldots, \boldsymbol{y}_{n_1^{(\ell)}}^{(\ell)} \overset{i.i.d.}{\sim} N_p(\boldsymbol{\eta}^{(\ell)}, \boldsymbol{\Delta}),\ \ \ell = 1, 2, \ldots, m$, where

$$\boldsymbol{\eta}^{(\ell)} = \begin{pmatrix} \boldsymbol{\eta}_1^{(\ell)} \\ \boldsymbol{\eta}_2^{(\ell)} \end{pmatrix} \begin{matrix} \}r \\ \}s \end{matrix}, \quad \boldsymbol{\Delta} = \left(\begin{array}{c|c} \overbrace{\boldsymbol{\Delta}_{11}}^{r} & \overbrace{\boldsymbol{\Delta}_{12}}^{s} \\ \hline \boldsymbol{\Delta}_{21} & \boldsymbol{\Delta}_{22} \end{array} \right) \begin{matrix} \}r \\ \}s \end{matrix},$$

with $r = p_2$, $s = p_1 - p_2$ (see Figure 2).

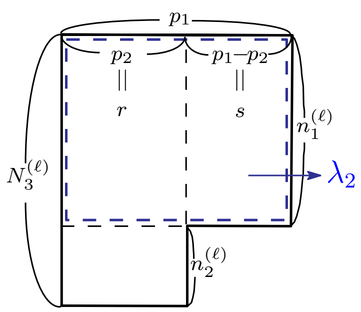

Figure 2. The LR λ_2 with two-step monotone missing data.

Then, because λ_2 is equal to the LR when testing

$$H_{02} : \boldsymbol{\eta}_2^{(1)} = \boldsymbol{\eta}_2^{(2)} = \cdots = \boldsymbol{\eta}_2^{(m)} \text{ given } \boldsymbol{\eta}_1^{(1)} = \boldsymbol{\eta}_1^{(2)} = \cdots = \boldsymbol{\eta}_1^{(m)}$$
$$\text{vs. } H_{12} : \boldsymbol{\eta}_2^{(i)} \neq \boldsymbol{\eta}_2^{(j)} \text{ for some } i \neq j \text{ given } \boldsymbol{\eta}_1^{(1)} = \boldsymbol{\eta}_1^{(2)} = \cdots = \boldsymbol{\eta}_1^{(m)},$$

we can write

$$\lambda_2 = \left(\frac{|\boldsymbol{I}_p + \boldsymbol{W}^{-1}\boldsymbol{B}|}{|\boldsymbol{I}_r + \boldsymbol{W}_{11}^{-1}\boldsymbol{B}_{11}|} \right)^{-\frac{M_2}{2}},$$

where

$$\boldsymbol{W} = \sum_{\ell=1}^{m} \boldsymbol{W}^{(\ell)}, \quad \boldsymbol{W}^{(\ell)} = \sum_{j=1}^{n_1^{(\ell)}} (\boldsymbol{y}_j^{(\ell)} - \overline{\boldsymbol{y}}^{(\ell)})(\boldsymbol{y}_j^{(\ell)} - \overline{\boldsymbol{y}}^{(\ell)})', \quad \overline{\boldsymbol{y}}^{(\ell)} = \frac{1}{n_1^{(\ell)}} \sum_{j=1}^{n_1^{(\ell)}} \boldsymbol{y}_j^{(\ell)},$$
$$\boldsymbol{B} = \sum_{\ell=1}^{m} n_1^{(\ell)} (\overline{\boldsymbol{y}}^{(\ell)} - \overline{\overline{\boldsymbol{y}}})(\overline{\boldsymbol{y}}^{(\ell)} - \overline{\overline{\boldsymbol{y}}})', \quad \overline{\overline{\boldsymbol{y}}} = \frac{1}{M_2} \sum_{\ell=1}^{m} n_1^{(\ell)} \overline{\boldsymbol{y}}^{(\ell)},$$

and

$$\overline{\boldsymbol{y}}^{(\ell)} = \begin{pmatrix} \overline{\boldsymbol{y}}_1^{(\ell)} \\ \overline{\boldsymbol{y}}_2^{(\ell)} \end{pmatrix} \begin{matrix} \}_r \\ \}_s \end{matrix}, \quad \boldsymbol{W} = \left(\begin{array}{c|c} \overbrace{\boldsymbol{W}_{11}}^{r} & \overbrace{\boldsymbol{W}_{12}}^{s} \\ \hline \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{array} \right) \begin{matrix} \}_r \\ \}_s \end{matrix}, \quad \boldsymbol{B} = \left(\begin{array}{c|c} \overbrace{\boldsymbol{B}_{11}}^{r} & \overbrace{\boldsymbol{B}_{12}}^{s} \\ \hline \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{array} \right) \begin{matrix} \}_r \\ \}_s \end{matrix}.$$

Without loss of generality, we may assume that $\boldsymbol{\Delta} = \boldsymbol{I}$. Therefore, under H_{02}, let $\boldsymbol{\eta}^{(1)} = \boldsymbol{\eta}^{(2)} = \cdots = \boldsymbol{\eta}^{(m)} = \boldsymbol{\eta}$, and let

$$\overline{\boldsymbol{y}}^{(\ell)} = \boldsymbol{\eta} + \frac{1}{\sqrt{n_1^{(\ell)}}} \boldsymbol{z}^{(\ell)}, \quad \frac{1}{n_1^{(\ell)} - 1} \boldsymbol{W}^{(\ell)} = \boldsymbol{I} + \frac{1}{\sqrt{n_1^{(\ell)}}} \boldsymbol{V}^{(\ell)}, \quad \ell = 1, 2, \ldots, m.$$

Furthermore, using the partitions of $\boldsymbol{z}^{(\ell)}$ and $\boldsymbol{V}^{(\ell)}$ as

$$\boldsymbol{z}^{(\ell)} = \begin{pmatrix} \boldsymbol{z}_1^{(\ell)} \\ \boldsymbol{z}_2^{(\ell)} \end{pmatrix} \begin{matrix} \}_r \\ \}_s \end{matrix}, \quad \boldsymbol{V}^{(\ell)} = \left(\begin{array}{c|c} \overbrace{\boldsymbol{V}_{11}^{(\ell)}}^{r} & \overbrace{\boldsymbol{V}_{12}^{(\ell)}}^{s} \\ \hline \boldsymbol{V}_{21}^{(\ell)} & \boldsymbol{V}_{22}^{(\ell)} \end{array} \right) \begin{matrix} \}_r \\ \}_s \end{matrix}, \quad \ell = 1, 2, \ldots, m,$$

we can expand Q_2 as

$$Q_2 = \mathrm{tr}\boldsymbol{B}_{22} + \frac{1}{\sqrt{n_1^{(1)}}} C_1 + \frac{1}{n_1^{(1)}} C_2 + O_p((n_1^{(1)})^{-\frac{3}{2}}),$$

where

$$C_1 = -\frac{1}{q}\mathrm{tr}\Big(\sum_{\ell=1}^{m}\sqrt{q_\ell}\boldsymbol{V}^{(\ell)}\Big)\boldsymbol{B} + \frac{1}{q}\mathrm{tr}\Big(\sum_{\ell=1}^{m}\sqrt{q_\ell}\boldsymbol{V}_{11}^{(\ell)}\Big)\boldsymbol{B}_{11},$$

$$C_2 = \frac{m}{q}\mathrm{tr}\boldsymbol{B}_{22} + \frac{1}{q^2}\Big\{\mathrm{tr}\Big(\sum_{\ell=1}^{m}\sqrt{q_\ell}\boldsymbol{V}^{(\ell)}\Big)^2\boldsymbol{B} - \mathrm{tr}\Big(\sum_{\ell=1}^{m}\sqrt{q_\ell}\boldsymbol{V}_{11}^{(\ell)}\Big)^2\boldsymbol{B}_{11}\Big\} - \frac{1}{2q}(\mathrm{tr}\boldsymbol{B}^2 - \mathrm{tr}\boldsymbol{B}_{11}^2),$$

$$\boldsymbol{B} = \sum_{\ell=1}^{m}\Big(1 - \frac{q_\ell}{q}\Big)\boldsymbol{z}^{(\ell)}\boldsymbol{z}^{(\ell)\prime} - \frac{1}{q}\sum_{\substack{i=1 \\ i\neq j}}^{m}\sum_{j=1}^{m}\sqrt{q_i q_j}\boldsymbol{z}^{(i)}\boldsymbol{z}^{(j)\prime},$$

$$q = \sum_{\ell=1}^{m} q_\ell, \quad q_\ell = \frac{n_1^{(\ell)}}{n_1^{(1)}}, \quad \ell = 1, 2, \ldots, m.$$

Hence,

$$\begin{aligned}\phi(t) =& E[\exp\{it(Q_2)\}] \\ =& E[\exp\{it(\mathrm{tr}\boldsymbol{B}_{22})\}] + E\left[\left(D + \frac{1}{2}D^2\right)\exp\{it(\mathrm{tr}\boldsymbol{B}_{22})\}\right] + O((n_1^{(1)})^{-\frac{3}{2}}),\end{aligned}$$

where

$$D = it\left(\frac{1}{\sqrt{n_1^{(1)}}}C_1 + \frac{1}{n_1^{(1)}}C_2\right).$$

After calculating the expectation and inverting the characteristic function, we obtain the following result:

$$\Pr(Q_2 \leq x) = G_{(m-1)s}(x) + \frac{\beta}{M_2}\left[G_{(m-1)s}(x) - G_{(m-1)s+2}(x)\right] + O(M_2^{-2}),$$

where $\beta = -(m-1)s(2r + s + m + 2)/4$. In addition, letting

$$\rho_2 = 1 - \frac{1}{2M_2}(p_1 + p_2 + m + 2),$$

the distribution function of $Q_2^*(= -2\rho_2 \log \lambda_2)$ is given by

$$\Pr(Q_2^* \leq x) = G_{(m-1)(p_1-p_2)}(x) + O(M_2^{-2}).$$

On the other hand, gathering up the expanded results for the characteristic function of Q_i, we can propose

$$Q^\dagger = -2\rho \log \lambda$$

as another MLRT statistic, where

$$\rho = 1 - \frac{1}{2p}\sum_{i=1}^{k}\frac{1}{M_{k-i+2}}(p_{k-i+1} - p_{k-i+2})(p_{k-i+1} + p_{k-i+2} + m + 2),$$

and $p_{k+1} = 0$. Note that the null distribution of $Q^\dagger$ can be expressed as

$$\Pr(Q^\dagger \leq x) = G_{(m-1)p}(x) + O(M_2^{-2}).$$

This means that the error from using the $\chi^2_{(m-1)p}$ distribution is of order M_2^{-2}.

4 Simulation studies

In this section, we investigate the numerical accuracy and the asymptotic behavior of the upper percentiles of the MLRT statistics using the actual type-I error rates. We compute the upper percentiles of the null distribution of the MLRT statistics in a one-way MANOVA using a Monte Carlo simulation (10^6 runs). That is, the LRT statistic Q and the MLRT statistics, Q^* and $Q^\dagger$, are computed 10^6 times, based on the normal random vectors generated from $N_{p_i}(\mathbf{0}, \boldsymbol{I}_{p_i})$ $i = 1, 2, \ldots, k$. In Tables 1–4, we provide the simulated upper 100α percentiles of Q, Q^*, and $Q^\dagger$, and their actual type-I error rates, α_q, α_{q^*}, and $\alpha_{q^\dagger}$, for the two-step case, respectively. Note that the actual type-I error rates are defined as

$$\alpha_q = \Pr\{Q > \chi^2_{(m-1)p}(\alpha)\}, \;\; \alpha_{q^*} = \Pr\{Q^* > \chi^2_{(m-1)p}(\alpha)\}, \;\; \text{and} \;\; \alpha_{q^\dagger} = \Pr\{Q^\dagger > \chi^2_{(m-1)p}(\alpha)\},$$

where $\chi^2_{(m-1)p}(\alpha)$ is the upper 100α percentile of the χ^2 distribution with $(m-1)p$ degrees of freedom. Computations are carried out for the following parameter sets, where $n_i^{(1)} = n_i^{(2)} = n_i^{(3)} = n_i$, $i = 1, 2, \ldots, k$:

$$\text{(I)} \quad k = 2;\; (p_1, p_2) = (8, 2), (8, 4), (15, 3), (15, 12);$$
$$n_1 = 10, 20, 50, 100, 200;\; n_2 = 5, 10, 50;\; \alpha = 0.05, 0.01.$$

Note that from Tables 1–4, that the value of $q(\alpha)$ converges very slowly to that of $\chi^2_{(m-1)p}(\alpha)$. However, the values of $q^*(\alpha)$ and $q^\dagger(\alpha)$ are closer to the upper percentiles of the χ^2 distribution with $(m-1)p$ degrees of freedom when the sample size n_1 becomes large. In particular, it can be seen that $q^*(\alpha)$ is a considerably good approximate value, even if the sample size n_1 is small. In addition, from Tables, the value of $q^*(\alpha)$ is closer to $\chi^2_{(m-1)p}(\alpha)$ when the value of p_2 is large. For example, comparing the cases of $(p_1, p_2) = (15, 3)$ and $(p_1, p_2) = (15, 12)$, the simulated values of $(p_1, p_2) = (15, 12)$ converge to the χ^2 distribution much faster than those of $(p_1, p_2) = (15, 3)$ do. Tables 5 and 6 and Tables 7 and 8 give the simulated results for the three-step case (II) and the five-step case (III), respectively:

(II) $k = 3$; $(p_1, p_2, p_3) = (15, 6, 3), (15, 12, 9)$;

$n_1 = 10, 20, 50, 100, 200$; $n_2 = n_3 = 5, 10, 50$; $\alpha = 0.05, 0.01$.

(III) $k = 5$; $(p_1, p_2, p_3, p_4, p_5) = (12, 8, 6, 4, 2), (15, 12, 9, 6, 3)$;

$n_1 = 10, 20, 50, 100, 200$, $n_2 = n_3 = \cdots = n_5 = 5, 10, 50$; $\alpha = 0.05, 0.01$.

From Tables 5–8, the asymptotic behavior of the approximations of the χ^2 distribution in the case of three-step or five-step monotone missing data show the same tendencies as in the two-step case.

In conclusion, we have developed the MLRT statistics Q^* and $Q^\dagger$ with general monotone missing data in a one-way MANOVA. Furthermore, we showed that the LR for this test can be decomposed into independent LRs of the test for a one-way MANOVA and those of a subvector with complete data. We also derived the asymptotic expansion using the perturbation procedure. The null distribution of our MLRT statistics is considerably closer to the χ^2 distribution than that of the LRT statistic, and we recommend the use of the MLRT statistic Q^* when the sample size is small.

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Table 1 : The upper percentiles of $Q(= -2\log\lambda)$, $Q^*(= \sum_{i=1}^{2} Q_i^*)$, $Q^\dagger(= -2\rho\log\lambda)$ and the actual type-I error rates when $(p_1, p_2) = (8, 2)$.

Sample Size		Upper Percentile			Type-I Error Rate		
n_1	n_2	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_q	α_{q^*}	$\alpha_{q^\dagger}$
			$\alpha = 0.05$				
10	5	33.69	26.45	26.72	0.195	0.052	0.056
20	5	29.49	26.34	26.38	0.102	0.051	0.051
50	5	27.48	26.30	26.30	0.067	0.050	0.050
100	5	26.84	26.27	26.27	0.058	0.050	0.050
200	5	26.58	26.29	26.29	0.054	0.050	0.050
10	10	33.52	26.42	26.75	0.191	0.052	0.056
20	10	29.39	26.30	26.35	0.100	0.050	0.051
50	10	27.44	26.28	26.28	0.067	0.050	0.050
100	10	26.88	26.30	26.31	0.058	0.050	0.050
200	10	26.58	26.30	26.29	0.054	0.050	0.050
10	50	33.29	26.44	26.89	0.185	0.052	0.058
20	50	29.25	26.31	26.39	0.098	0.050	0.051
50	50	27.44	26.32	26.33	0.066	0.050	0.050
100	50	26.86	26.30	26.30	0.058	0.050	0.050
200	50	26.57	26.29	26.29	0.054	0.050	0.050
			$\alpha = 0.01$				
10	5	41.11	32.23	32.61	0.070	0.011	0.012
20	5	35.85	31.98	32.07	0.027	0.010	0.010
50	5	33.43	32.00	32.00	0.015	0.010	0.010
100	5	32.68	31.99	31.98	0.012	0.010	0.010
200	5	32.38	32.03	32.03	0.011	0.010	0.010
10	10	40.82	32.17	32.57	0.068	0.010	0.012
20	10	35.82	32.06	32.11	0.027	0.010	0.010
50	10	33.44	32.01	32.02	0.015	0.010	0.010
100	10	32.75	32.04	32.04	0.012	0.010	0.010
200	10	32.35	32.00	32.00	0.011	0.010	0.010
10	50	40.59	32.19	32.78	0.065	0.011	0.013
20	50	35.65	32.05	32.16	0.026	0.010	0.010
50	50	33.37	32.01	32.02	0.015	0.010	0.010
100	50	32.73	32.06	32.06	0.012	0.010	0.010
200	50	32.29	31.95	31.95	0.011	0.010	0.010

Note. $\chi^2_{16}(0.05) = 26.30$, $\chi^2_{16}(0.01) = 32.00$.

Table 2 : The upper percentiles of $Q(=-2\log\lambda)$, $Q^*(=\sum_{i=1}^{2} Q_i^*)$, $Q^\dagger(=-2\rho\log\lambda)$ and the actual type-I error rates when $(p_1, p_2) = (8, 4)$.

Sample Size		Upper Percentile			Type-I Error Rate		
n_1	n_2	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_q	α_{q^*}	$\alpha_{q^\dagger}$
			$\alpha = 0.05$				
10	5	33.15	26.36	26.80	0.181	0.051	0.057
20	5	29.32	26.30	26.37	0.099	0.050	0.051
50	5	27.44	26.28	26.29	0.066	0.050	0.050
100	5	26.87	26.30	26.30	0.058	0.050	0.050
200	5	26.60	26.31	26.31	0.054	0.050	0.050
10	10	32.73	26.33	26.87	0.173	0.050	0.058
20	10	29.18	26.31	26.38	0.096	0.050	0.051
50	10	27.41	26.29	26.29	0.066	0.050	0.050
100	10	26.85	26.29	26.29	0.058	0.050	0.050
200	10	26.57	26.29	26.29	0.054	0.050	0.050
10	50	32.05	26.33	27.11	0.155	0.050	0.061
20	50	28.79	26.31	26.44	0.089	0.050	0.052
50	50	27.29	26.29	26.31	0.064	0.050	0.050
100	50	26.81	26.30	26.30	0.057	0.050	0.050
200	50	26.57	26.31	26.31	0.054	0.050	0.050
			$\alpha = 0.01$				
10	5	40.41	32.09	32.66	0.063	0.010	0.012
20	5	35.72	32.02	32.12	0.026	0.010	0.010
50	5	33.44	32.04	32.04	0.015	0.010	0.010
100	5	32.68	31.98	31.98	0.012	0.010	0.010
200	5	32.34	31.98	31.99	0.011	0.010	0.010
10	10	39.93	32.03	32.78	0.058	0.010	0.013
20	10	35.53	32.02	32.13	0.025	0.010	0.010
50	10	33.36	31.99	32.00	0.015	0.010	0.010
100	10	32.66	31.97	31.97	0.012	0.010	0.010
200	10	32.38	32.03	32.04	0.011	0.010	0.010
10	50	39.12	32.03	33.09	0.051	0.010	0.014
20	50	35.06	32.01	32.20	0.022	0.010	0.011
50	50	33.21	32.00	32.02	0.014	0.010	0.010
100	50	32.59	31.97	31.97	0.012	0.010	0.010
200	50	32.33	32.00	32.00	0.011	0.010	0.010

Note. $\chi^2_{16}(0.05) = 26.30$, $\chi^2_{16}(0.01) = 32.00$.

Table 3 : The upper percentiles of $Q(= -2\log\lambda)$, $Q^*(= \sum_{i=1}^{2} Q_i^*)$, $Q^\dagger(= -2\rho\log\lambda)$ and the actual type-I error rates when $(p_1, p_2) = (15, 3)$.

Sample Size		Upper Percentile			Type-I Error Rate		
n_1	n_2	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_q	α_{q^*}	$\alpha_{q^\dagger}$
			$\alpha = 0.05$				
10	5	69.23	45.33	46.77	0.567	0.068	0.086
20	5	52.88	43.99	44.21	0.199	0.052	0.055
50	5	46.95	43.81	43.84	0.090	0.050	0.051
100	5	45.30	43.79	43.80	0.068	0.050	0.050
200	5	44.51	43.77	43.77	0.058	0.050	0.050
10	10	69.01	45.31	46.92	0.562	0.067	0.088
20	10	52.78	43.97	44.22	0.198	0.052	0.055
50	10	46.92	43.81	43.83	0.090	0.050	0.051
100	10	45.29	43.78	43.79	0.067	0.050	0.050
200	10	44.51	43.77	43.77	0.058	0.050	0.050
10	50	68.64	45.33	47.28	0.552	0.067	0.093
20	50	52.61	44.03	44.34	0.193	0.053	0.056
50	50	46.78	43.75	43.78	0.088	0.050	0.050
100	50	45.27	43.80	43.80	0.067	0.050	0.050
200	50	44.50	43.77	43.77	0.058	0.050	0.050
			$\alpha = 0.01$				
10	5	80.93	52.84	54.67	0.345	0.016	0.022
20	5	61.56	51.19	51.46	0.069	0.011	0.011
50	5	54.58	50.93	50.97	0.022	0.010	0.010
100	5	52.67	50.92	50.92	0.015	0.010	0.010
200	5	51.76	50.90	50.90	0.012	0.010	0.010
10	10	80.61	52.77	54.82	0.340	0.015	0.023
20	10	61.44	51.16	51.47	0.068	0.011	0.012
50	10	54.52	50.88	50.94	0.022	0.010	0.010
100	10	52.61	50.86	50.87	0.015	0.010	0.010
200	10	51.77	50.91	50.91	0.012	0.010	0.010
10	50	80.41	52.91	55.39	0.331	0.016	0.025
20	50	61.15	51.13	51.54	0.067	0.011	0.012
50	50	54.49	50.97	51.01	0.022	0.010	0.010
100	50	52.67	50.94	50.96	0.015	0.010	0.010
200	50	51.79	50.95	50.95	0.012	0.010	0.010

Note. $\chi^2_{30}(0.05) = 43.77$, $\chi^2_{30}(0.01) = 50.89$.

Table 4 : The upper percentiles of $Q(=-2\log\lambda)$, $Q^*(=\sum_{i=1}^{2} Q_i^*)$, $Q^\dagger(=-2\rho\log\lambda)$ and the actual type-I error rates when $(p_1, p_2) = (15, 12)$.

Sample Size		Upper Percentile			Type-I Error Rate		
n_1	n_2	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_q	α_{q^*}	$\alpha_{q^\dagger}$
			$\alpha = 0.05$				
10	5	63.18	44.20	46.90	0.427	0.054	0.087
20	5	51.63	43.90	44.19	0.173	0.051	0.054
50	5	46.73	43.78	43.81	0.087	0.050	0.050
100	5	45.25	43.79	43.79	0.067	0.050	0.050
200	5	44.51	43.78	43.78	0.058	0.050	0.050
10	10	60.74	44.00	47.38	0.365	0.052	0.093
20	10	50.79	43.87	44.25	0.155	0.051	0.055
50	10	46.61	43.81	43.85	0.085	0.050	0.051
100	10	45.22	43.79	43.80	0.066	0.050	0.050
200	10	44.48	43.77	43.77	0.058	0.050	0.050
10	50	56.80	43.79	48.60	0.267	0.050	0.110
20	50	48.54	43.78	44.38	0.115	0.050	0.056
50	50	45.84	43.77	43.82	0.075	0.050	0.050
100	50	44.93	43.76	43.77	0.063	0.050	0.050
200	50	44.39	43.76	43.75	0.057	0.050	0.050
			$\alpha = 0.01$				
10	5	74.25	51.41	55.11	0.222	0.011	0.023
20	5	60.14	51.08	51.48	0.057	0.010	0.011
50	5	54.37	50.92	50.97	0.021	0.010	0.010
100	5	52.60	50.90	50.90	0.015	0.010	0.010
200	5	51.71	50.86	50.87	0.012	0.010	0.010
10	10	71.41	51.19	55.70	0.177	0.011	0.026
20	10	59.14	51.01	51.52	0.049	0.010	0.012
50	10	54.20	50.93	50.99	0.021	0.010	0.010
100	10	52.54	50.90	50.90	0.015	0.010	0.010
200	10	51.74	50.91	50.90	0.012	0.010	0.010
10	50	66.96	50.89	57.28	0.114	0.010	0.033
20	50	56.51	50.85	51.67	0.032	0.010	0.012
50	50	53.34	50.91	50.99	0.017	0.010	0.010
100	50	52.28	50.92	50.93	0.014	0.010	0.010
200	50	51.62	50.87	50.88	0.012	0.010	0.010

Note. $\chi^2_{30}(0.05) = 43.77$, $\chi^2_{30}(0.01) = 50.89$.

Table 5 : The upper percentiles of $Q(= -2\log\lambda)$, $Q^*(= \sum_{i=1}^{3} Q_i^*)$, $Q^\dagger(= -2\rho\log\lambda)$ and the actual type-I error rates when $(p_1, p_2, p_3) = (15, 6, 3)$.

Sample Size		Upper Percentile			Type-I Error Rate		
n_1	$n_2 = n_3$	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_q	α_{q^*}	$\alpha_{q^\dagger}$
			$\alpha = 0.05$				
10	5	68.00	44.53	47.30	0.538	0.058	0.093
20	5	52.54	43.87	44.26	0.192	0.051	0.055
50	5	46.86	43.77	43.81	0.089	0.050	0.050
100	5	45.26	43.76	43.77	0.067	0.050	0.050
200	5	44.52	43.78	43.78	0.058	0.050	0.050
10	10	67.43	44.53	47.72	0.523	0.058	0.099
20	10	52.33	43.90	44.37	0.187	0.051	0.056
50	10	46.84	43.82	43.86	0.088	0.050	0.051
100	10	45.21	43.73	43.74	0.066	0.050	0.050
200	10	44.48	43.75	43.75	0.058	0.050	0.050
10	50	66.46	44.53	48.50	0.498	0.058	0.111
20	50	51.67	43.88	44.49	0.174	0.051	0.058
50	50	46.59	43.81	43.87	0.085	0.050	0.051
100	50	45.16	43.77	43.79	0.066	0.050	0.050
200	50	44.47	43.77	43.77	0.058	0.050	0.050
			$\alpha = 0.01$				
10	5	79.64	51.84	55.39	0.318	0.012	0.025
20	5	61.17	51.05	51.53	0.066	0.010	0.012
50	5	54.50	50.90	50.96	0.022	0.010	0.010
100	5	52.59	50.86	50.86	0.015	0.010	0.010
200	5	51.77	50.91	50.91	0.012	0.010	0.010
10	10	79.06	51.85	55.96	0.305	0.012	0.028
20	10	60.86	51.05	51.60	0.064	0.010	0.012
50	10	54.45	50.95	50.99	0.022	0.010	0.010
100	10	52.59	50.88	50.88	0.015	0.010	0.010
200	10	51.68	50.83	50.83	0.012	0.010	0.010
10	50	77.89	51.88	56.84	0.283	0.012	0.032
20	50	60.19	51.01	51.83	0.057	0.010	0.012
50	50	54.15	50.89	50.99	0.021	0.010	0.010
100	50	52.47	50.85	50.87	0.015	0.010	0.010
200	50	51.68	50.86	50.87	0.012	0.010	0.010

Note. $\chi^2_{30}(0.05) = 43.77$, $\chi^2_{30}(0.01) = 50.89$.

Table 6 : The upper percentiles of $Q(= -2 \log \lambda)$, $Q^*(= \sum_{i=1}^{3} Q_i^*)$, $Q^\dagger(= -2\rho \log \lambda)$ and the actual type-I error rates when $(p_1, p_2, p_3) = (15, 12, 9)$.

Sample Size		Upper Percentile			Type-I Error Rate		
n_1	$n_2 = n_3$	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_q	α_{q^*}	$\alpha_{q^\dagger}$
			$\alpha = 0.05$				
10	5	61.90	43.91	47.39	0.393	0.051	0.093
20	5	51.12	43.80	44.23	0.162	0.050	0.055
50	5	46.63	43.77	43.81	0.085	0.050	0.050
100	5	45.21	43.77	43.78	0.066	0.050	0.050
200	5	44.49	43.77	43.77	0.058	0.050	0.050
10	10	59.59	43.85	47.87	0.334	0.051	0.100
20	10	50.19	43.82	44.31	0.144	0.050	0.056
50	10	46.41	43.77	43.82	0.082	0.050	0.050
100	10	45.14	43.77	43.77	0.065	0.050	0.050
200	10	44.48	43.77	43.77	0.058	0.050	0.050
10	50	56.34	43.78	48.80	0.256	0.050	0.113
20	50	48.19	43.77	44.46	0.109	0.050	0.057
50	50	45.67	43.80	43.87	0.072	0.050	0.051
100	50	44.88	43.81	43.83	0.062	0.050	0.051
200	50	44.39	43.79	43.79	0.057	0.050	0.050
			$\alpha = 0.01$				
10	5	72.67	51.08	55.63	0.196	0.010	0.026
20	5	59.49	50.93	51.48	0.052	0.010	0.012
50	5	54.24	50.91	50.96	0.021	0.010	0.010
100	5	52.60	50.93	50.94	0.015	0.010	0.010
200	5	51.78	50.95	50.94	0.012	0.010	0.010
10	10	70.07	51.02	56.29	0.156	0.010	0.029
20	10	58.40	50.95	51.55	0.044	0.010	0.012
50	10	54.01	50.94	51.00	0.020	0.010	0.010
100	10	52.42	50.83	50.84	0.014	0.010	0.010
200	10	51.71	50.89	50.89	0.012	0.010	0.010
10	50	66.64	50.98	57.72	0.108	0.010	0.035
20	50	56.15	50.94	51.80	0.030	0.010	0.012
50	50	53.13	50.94	51.04	0.017	0.010	0.010
100	50	52.18	50.93	50.96	0.013	0.010	0.010
200	50	51.60	50.90	50.90	0.012	0.010	0.010

Note. $\chi^2_{30}(0.05) = 43.77$, $\chi^2_{30}(0.01) = 50.89$.

Table 7 : The upper percentiles of $Q(= -2\log\lambda)$, $Q^*(= \sum_{i=1}^{5} Q_i^*)$, $Q^\dagger(= -2\rho\log\lambda)$ and the actual type-I error rates when $(p_1, p_2, p_3, p_4, p_5) = (12, 8, 6, 4, 2)$.

Sample Size		Upper Percentile			Type-I Error Rate		
n_1	$n_2 = \cdots = n_5$	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_q	α_{q^*}	$\alpha_{q^\dagger}$
			$\alpha = 0.05$				
10	5	48.86	36.47	38.42	0.288	0.051	0.075
20	5	41.64	36.44	36.73	0.130	0.050	0.054
50	5	38.42	36.40	36.43	0.076	0.050	0.050
100	5	37.42	36.40	36.41	0.062	0.050	0.050
200	5	36.93	36.42	36.42	0.056	0.050	0.050
10	10	47.80	36.45	38.69	0.262	0.050	0.079
20	10	41.16	36.42	36.76	0.121	0.050	0.054
50	10	38.25	36.35	36.39	0.074	0.049	0.050
100	10	37.39	36.41	36.42	0.062	0.050	0.050
200	10	36.92	36.42	36.42	0.056	0.050	0.050
10	50	46.44	36.49	39.27	0.225	0.051	0.087
20	50	40.23	36.43	36.89	0.104	0.050	0.055
50	50	37.90	36.40	36.45	0.069	0.050	0.050
100	50	37.22	36.41	36.42	0.060	0.050	0.050
200	50	36.84	36.40	36.40	0.055	0.050	0.050
			$\alpha = 0.01$				
10	5	57.94	43.02	45.56	0.123	0.010	0.018
20	5	49.15	42.99	43.35	0.038	0.010	0.011
50	5	45.40	43.01	43.05	0.018	0.010	0.010
100	5	44.21	43.01	43.01	0.014	0.010	0.010
200	5	43.57	42.96	42.96	0.012	0.010	0.010
10	10	56.85	43.04	46.01	0.107	0.010	0.020
20	10	48.62	42.98	43.42	0.034	0.010	0.011
50	10	45.11	42.89	42.93	0.017	0.010	0.010
100	10	44.11	42.95	42.97	0.013	0.010	0.010
200	10	43.53	42.94	42.95	0.012	0.010	0.010
10	50	55.28	43.09	46.75	0.088	0.010	0.023
20	50	47.55	42.99	43.60	0.028	0.010	0.012
50	50	44.76	42.98	43.05	0.016	0.010	0.010
100	50	43.97	43.03	43.03	0.013	0.010	0.010
200	50	43.45	42.93	42.94	0.011	0.010	0.010

Note. $\chi^2_{24}(0.05) = 36.42$, $\chi^2_{24}(0.01) = 42.98$.

Table 8 : The upper percentiles of $Q(=-2\log\lambda)$, $Q^*(=\sum_{i=1}^{5} Q_i^*)$, $Q^\dagger(=-2\rho\log\lambda)$ and the actual type-I error rates when $(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3)$.

Sample Size		Upper Percentile			Type-I Error Rate		
n_1	$n_2 = \cdots = n_5$	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_q	α_{q^*}	$\alpha_{q^\dagger}$
			$\alpha = 0.05$				
10	5	61.39	43.80	47.56	0.381	0.050	0.096
20	5	50.91	43.76	44.28	0.159	0.050	0.056
50	5	46.60	43.78	43.84	0.085	0.050	0.051
100	5	45.22	43.79	43.80	0.066	0.050	0.050
200	5	44.50	43.78	43.78	0.058	0.050	0.050
10	10	59.18	43.81	47.98	0.326	0.050	0.102
20	10	49.93	43.76	44.30	0.139	0.050	0.056
50	10	46.31	43.74	43.81	0.081	0.050	0.050
100	10	45.12	43.77	43.79	0.065	0.050	0.050
200	10	44.47	43.77	43.78	0.058	0.050	0.050
10	50	56.29	43.81	48.90	0.254	0.050	0.114
20	50	48.10	43.79	44.49	0.107	0.050	0.058
50	50	45.61	43.81	43.89	0.071	0.050	0.051
100	50	44.78	43.76	43.77	0.061	0.050	0.050
200	50	44.36	43.78	43.78	0.056	0.050	0.050
			$\alpha = 0.01$				
10	5	72.18	50.99	55.92	0.188	0.010	0.027
20	5	59.17	50.86	51.47	0.050	0.010	0.012
50	5	54.17	50.90	50.97	0.021	0.010	0.010
100	5	52.51	50.86	50.86	0.015	0.010	0.010
200	5	51.80	50.96	50.96	0.012	0.010	0.010
10	10	69.67	50.95	56.48	0.150	0.010	0.030
20	10	58.12	50.86	51.57	0.042	0.010	0.012
50	10	53.84	50.84	50.93	0.019	0.010	0.010
100	10	52.50	50.92	50.95	0.015	0.010	0.010
200	10	51.70	50.89	50.89	0.012	0.010	0.010
10	50	66.48	50.95	57.75	0.107	0.010	0.035
20	50	56.01	50.90	51.80	0.029	0.010	0.012
50	50	53.07	50.96	51.06	0.016	0.010	0.010
100	50	52.06	50.86	50.88	0.013	0.010	0.010
200	50	51.60	50.94	50.93	0.012	0.010	0.010

Note. $\chi^2_{30}(0.05) = 43.77$, $\chi^2_{30}(0.01) = 50.89$.