High-Dimensional Asymptotic Distributions of Simplified MLEs in Growth Curve Model with an Autoregressive Covariance Structure

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Abstract

In this paper, we consider the growth curve model with an autregressive covariance structure. The purpose of this paper is to derive high-dimensional asymptotic distributions of the simplified MLEs. High-dimensional asymptotic distributions are also given for some basic statistics. Accuracies of the asymptotic distributions are examined through simulation experiments.

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1. Introduction

The growth curve model introduced by Potthoff and Roy (1964) is written as

$$\mathbf{Y} = \mathbf{A}\boldsymbol{\Theta}\mathbf{X} + \boldsymbol{\mathcal{E}}, \tag{1.1}$$

where $\mathbf{Y}; n \times p$ is an observation matrix, $\mathbf{A}; n \times q$ is a design matrix across individuals, $\mathbf{X}; k \times p$ is a design matrix within individuals, $\boldsymbol{\Theta}; q \times k$ is an unknown matrix, and each row of $\boldsymbol{\mathcal{E}}; n \times p$ is independent and identically distributed as a p-dimensional normal distribution with mean $\mathbf{0}$ and an unknown covariance matrix $\boldsymbol{\Sigma}$. We assume that $n - p - k - 1 > 0$ and $\text{rank}(\mathbf{X}) = k$.

In this paper we assume that the covariance matrix has an autoregressive structure given by

$$\boldsymbol{\Sigma} = \sigma^2(\rho^{|i-j|})_{1 \le i,j \le p}. \tag{1.2}$$

The purpose of this paper is to derive asymptotic distributions of the simplified MLEs when the sample size n and the number p of repeated measurements are large, satisfying $p/n \to c \in [0, 1)$. High-dimensional asymptotic distributions are given for some basic statistics. Accuracies of our asymptotic distributions are examined through simulation experiments.

The present paper is organized as follows. In section 2, we present simplified MLEs. In Section 3, High-dimensional asymptotic distributions are derived. In Section 4 numerical accuracies are dstudied. The proofs of our asymptotic results is given in Appendix.

2. Simplified MLEs

It is known (see, e.g., Fujikoshi et al. (1990)) that the MLEs $\widetilde{\Theta}$, $\widetilde{\sigma}^2$ and $\widetilde{\rho}$ of Θ, σ^2 and ρ are given as the solutions of the following simultaneos equations:

$$
\begin{aligned}
&(1)\quad \boldsymbol{\Theta} = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{Y}\mathbf{G}^{-1}\mathbf{X}'(\mathbf{X}\mathbf{G}^{-1}\mathbf{X}')^{-1},\\
&(2)\quad \sigma^2 = \frac{n}{N}\frac{1}{p(1-\rho^2)}(\widetilde{a}_1\rho^2 - 2\widetilde{a}_2\rho + \widetilde{a}_0),\\
&(3)\quad (p-1)\tilde{a}_1\rho^3 - (p-2)\widetilde{a}_2\rho^2 - (p\widetilde{a}_1 + \widetilde{a}_0)\rho + p\widetilde{a}_2 = 0.
\end{aligned}
$$

Here, $\mathbf{G} = (\rho^{|i-j|})_{1\le i,j\le p} : p \times p, \quad \widetilde{a}_i = \mathrm{tr}\mathbf{C}_i\mathbf{R}, \quad i = 0, 1, 2,$

$$
\mathbf{R} = \frac{1}{n}(\mathbf{Y} - \mathbf{A}\boldsymbol{\Theta}\mathbf{X})(\mathbf{Y} - \mathbf{A}\boldsymbol{\Theta}\mathbf{X}), \quad n = N - q.
$$

$$
\mathbf{C}_0 = \mathbf{I}_p \quad \mathbf{C}_1 = \begin{pmatrix} 0 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 0 & 0 \end{pmatrix}, \quad \mathbf{C}_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & & 1 & 0 \end{pmatrix}.
$$

In this paper we consider a simplified MLE $\widehat{\rho}$ of ρ obtained by replacing $\tilde{a}_i$ in (3) with a_i,

$$
a_i = \mathrm{tr}\mathbf{C}_i\mathbf{S}, \quad i = 0, 1, 2, \quad \mathbf{S} = \frac{1}{n}\mathbf{Y}'(\mathbf{I}_n - \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}')\mathbf{Y},
$$

and $n = N - k$. That is, the simplified MLE $\widehat{\rho}$ of ρ is defined by the solution of

$$
(p-1)a_1\widehat{\rho}^3 - (p-2)a_2\widehat{\rho}^2 - (pa_1 + a_0)\widehat{\rho} + pa_2 = 0. \tag{2.1}
$$

Using the simplified MLE $\widehat{\rho}$, the simpified MLEs $\widehat{\boldsymbol{\Theta}}$ and $\widehat{\sigma}^2$ of $\boldsymbol{\Theta}$ and σ^2 are defined by

$$
\widehat{\boldsymbol{\Theta}} = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{Y}\widehat{\mathbf{G}}^{-1}\mathbf{X}'(\mathbf{X}\widehat{\mathbf{G}}^{-1}\mathbf{X}')^{-1}, \tag{2.2}
$$

$$
\widehat{\sigma}^2 = \frac{n}{N}\frac{1}{p(1-\widehat{\rho}^2)}(a_1\widehat{\rho}^2 - 2a_2\widehat{\rho} + a_0), \tag{2.3}
$$

where $\widehat{\mathbf{G}} = (\widehat{\rho}^{|i-j|})_{1\le i,j\le p} : p \times p$.

Note that when $\mathbf{X} = \mathbf{I}_p$, the simplified MLEs are coincident with the MLEs. Further, the simplified MLEs were used as an initial value for solving the simultaneous equations (1), (2) and (3), i.e., for obtaining the MLE under (1.1).

3. High-Dimensional Asymptotic Distributions

Our simplified MLEs $\widehat{\rho}$, $\widehat{\sigma}^2$ and $\widehat{\boldsymbol{\Theta}}$ of concern are defined through (2.1), (2.2) and (2.3), in terms of a_0, a_1 and a_2. Our main purpose is to derive asymptotic distributions of $\widehat{\rho}$, $\widehat{\sigma}^2$ and $\widehat{\Theta}$ under

$$\text{A1; } p/n \to c \in (0,1). \tag{3.1}$$

Based on asymptotic behaiviors of a_i's in Appendix, $\hat{\rho}$ may be regardes as the solution of

$$\begin{aligned} & a_1\rho^3 - a_2\rho^2 - a_1\rho + a_2 + \mathrm{O}(p^{-1}) = 0, \\ & \Leftrightarrow (a_1\rho - a_2)(\rho^2 - 1) + \mathrm{O}(p^{-1}) = 0, \end{aligned}$$

and hence

$$\widehat{\rho} = \frac{a_2}{a_1} + \mathrm{O}(p^{-1}). \tag{3.2}$$

Under (3.1) we may obtain an asymptotic distribution of $\widehat{\rho}$ by considering the one of a_2/a_1. Further, we can derive asymptotic distributions of $\widehat{\sigma}^2$ and $\widehat{\boldsymbol{\Theta}}$ by applying delta method to (2.3) and (2.2), respectively. The results are given in the following theorem whose derivation is given in Appendix.

Theorem 3.1. *Let $\widehat{\rho}$ and $\widehat{\sigma}^2$ be the simplefied MLEs defined by* (2.1) *and* (2.3)*, respectively. Then, under a high-dimensional asymptotic framework* (3.1) *it holds that*

$$\begin{aligned} & (1)\ n\left(\hat{\rho} - \frac{p-1}{p-2}\rho\right) \xrightarrow{d} \mathrm{N}\left(0, \frac{1}{c}(1-\rho^2)\right), \\ & (2)\ n\left(\hat{\sigma}^2 - \left\{\frac{p-2}{p} + \frac{2}{p}\frac{(p-2)^2}{(p-2)^2 - (p-1)^2\rho^2}\right\}\sigma^2\right) \\ & \qquad \xrightarrow{d} \mathrm{N}\left(0, 2\sigma^4\frac{(1+\rho^2)}{c(1-\rho^2)}\right), \end{aligned}$$

where $\xrightarrow{d}$ denotes the convergence in distribution.

From Theorem 3.1 (1) and (2) we have

$$\widehat{\rho} = \rho + \mathrm{O}(n^{-1}), \quad \widehat{\sigma}^2 = \sigma^2 + \mathrm{O}(n^{-1}),$$

and hence $\widehat{\rho}$ and $\widehat{\sigma}^2$ are consistent. Such properties in a high-dimensional framework have been also studied in Sakurai and Fujikoshi (2017).

Next we consider the distribution of a standardized estimator of $\widehat{\mathbf{\Theta}}$ defined by

$$\tilde{\mathbf{\Theta}} = (\mathbf{A}'\mathbf{A})^{1/2}(\widehat{\mathbf{\Theta}} - \mathbf{\Theta})(\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}')^{1/2}. \tag{3.3}$$

From (2.2) the standardized estimator is expressed as

$$\begin{aligned}\tilde{\mathbf{\Theta}} &= (\mathbf{A}'\mathbf{A})^{1/2}\left\{(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'(\mathsf{Y} - \mathbf{A}\mathbf{\Theta}\mathbf{X})\widehat{\mathbf{\Sigma}}^{-1}\mathbf{X}'(\mathbf{X}\widehat{\mathbf{\Sigma}}^{-1}\mathbf{X}')^{-1}\right\} \\ &\quad \times (\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}')^{1/2} \\ &= \mathsf{Z} + \mathsf{U}\mathsf{V},\end{aligned}$$

where $\widehat{\mathbf{\Sigma}} = \widehat{\sigma}^2(\widehat{\rho}^{|i-j|})_{1\le i,j\le 1}$,

$$\begin{aligned}\mathsf{Z} &= \mathsf{U}\mathbf{\Sigma}^{-1/2}\mathbf{X}'(\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}')^{-1/2}, \\ \mathsf{U} &= (\mathbf{A}'\mathbf{A})^{-1/2}\mathbf{A}'(\mathsf{Y} - \mathbf{A}\mathbf{\Theta}\mathbf{X})\mathbf{\Sigma}^{-1/2}, \\ \mathsf{V} &= \mathbf{\Sigma}^{1/2}\widehat{\mathbf{\Sigma}}^{-1}\mathbf{X}'(\mathbf{X}\widehat{\mathbf{\Sigma}}^{-1}\mathbf{X}')^{-1}(\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}')^{1/2} - \mathbf{\Sigma}^{-1/2}\mathbf{X}'(\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}')^{1/2}.\end{aligned}$$

The elements of $\mathsf{U}; q \times p$ are independently distributed as $\mathrm{N}(0, 1)$. Noting that

$$\left\{\mathbf{\Sigma}^{-1/2}\mathbf{X}'(\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}')^{1/2}\right\}'\mathbf{\Sigma}^{-1/2}\mathbf{X}'(\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}')^{1/2} = \mathbf{I}_q,$$

we have that the elements of $\mathsf{Z}; q\times k$ are independently distributed as $\mathrm{N}(0, 1)$.

Theorem 3.2. *Let $\widehat{\mathbf{\Theta}}$ be the simplefied MLE defined by* (2.2)*. Then, under a high-dimensional asymptotic framework* (3.1) *it holds that the elements of*

$$\tilde{\mathbf{\Theta}} = (\mathbf{A}'\mathbf{A})^{1/2}(\widehat{\mathbf{\Theta}} - \mathbf{\Theta})(\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}')^{1/2}$$

are asymptotically idependent and distributed as $\mathrm{N}(0, 1)$.

4. Numerical Accuracies

In this section, we numerically examine accuracies of the asymtptic distributions of $\widehat{\rho}$ and $\widehat{\sigma}^2$ given in Theorem 3.1. Our numerical experiments were done in the case of $q = 2$, $k = 3$, $n_1 = n_2 = n_0$, $\sigma^2 = 4$, and

$$\rho = 0.2, 0.8, \quad (N, p) = (2n_0, p) = (100, 50), (200, 100), (400, 200),$$

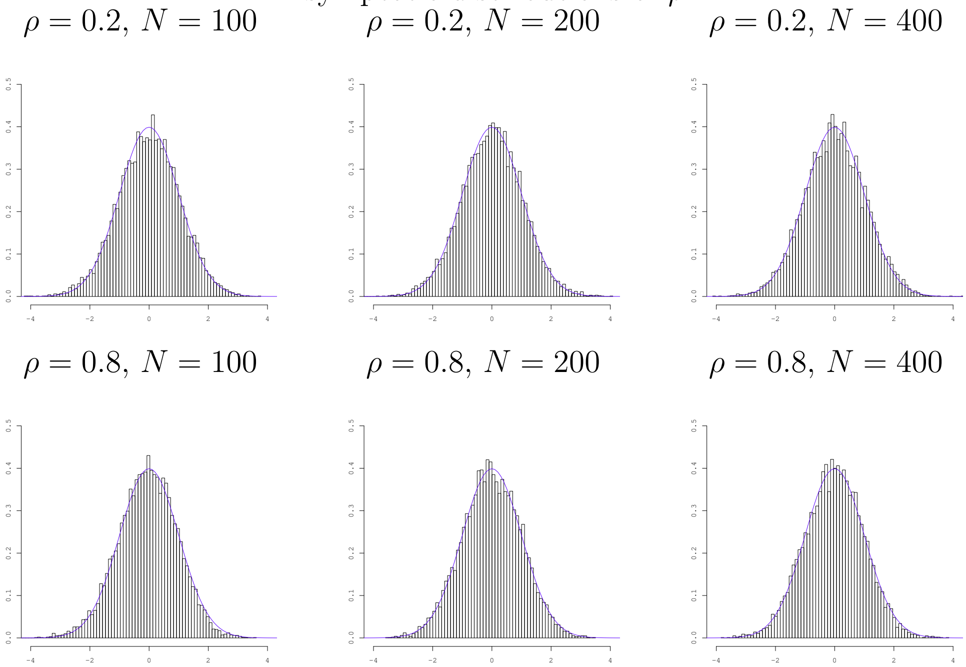
for Monte Carlo simulations with 10^4 replications. Asymptotic means and variances were compared with their simulation results in Table 4.1. It is seen that our asymptotic means and variances are very accurate.

Table 4.1. Aymptotic means and variances of $\widehat{\rho}$ and $\widehat{\sigma^2}$

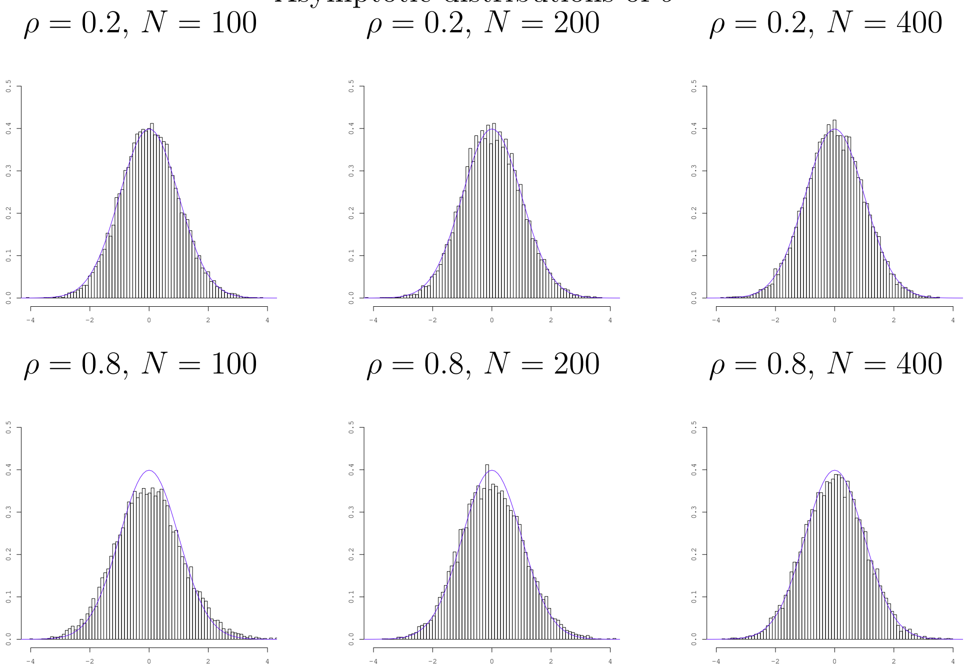
			$\hat{\rho}$				$\hat{\sigma}^2$			
			mean		variance		mean		variance	
ρ	N	p	sim	asy	sim	asy	sim	asy	sim	asy
	100	50	0.20	0.20	1.99	1.88	3.93	4.01	65.61	67.95
0.2	200	100	0.20	0.20	1.97	1.90	3.96	4.04	66.90	68.64
	400	200	0.20	0.20	1.98	1.91	3.98	4.06	68.44	68.99
	100	50	0.82	0.82	0.70	0.71	4.23	4.32	358.96	285.72
0.8	200	100	0.81	0.81	0.71	0.71	4.11	4.19	321.91	288.64
	400	200	0.80	0.80	0.71	0.72	4.05	4.14	303.56	290.10

Next accuracies of the asymptotic distributions themselves of $\widehat{\rho}$ and $\widehat{\sigma}^2$ were examined. Histgrams denote the simulation results.

Asymptotic distributions of $\hat{\rho}$



Asymptotic distributions of $\hat{\sigma}^2$



Appendix: The Proof of Theorem 3.1 (1) and (2)

A1. Asymptotic Distributions of $\mathrm{tr}\mathbf{A}\mathsf{W}$ and $\mathrm{tr}\mathbf{A}\mathsf{S}$

In this subsection we give asymptotic results on the distributions of $\mathrm{tr}\mathbf{A}\mathsf{W}$ and $\mathrm{tr}\mathbf{A}\mathsf{S}$, where

$$\mathbf{A}' = \mathbf{A}, \quad \mathsf{W} = n\mathsf{S} \sim \mathrm{W}_p(n, \boldsymbol{\Sigma}),$$

under

$$\text{A1}; \quad n, p \to \infty, \quad p/n \to c \in [0, 1). \tag{A1}$$

It is known (see, e.g., Fujikoshi et al. (2010)) that

$$\mathrm{E}(\mathrm{tr}\mathbf{A}\mathsf{W}) = n\mathrm{tr}\mathbf{A}\boldsymbol{\Sigma}, \quad \mathrm{Var}(\mathrm{tr}\mathbf{A}\mathsf{W}) = 2n\mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma})^2, \tag{A2}$$

$$\mathrm{E}(\mathrm{tr}\mathbf{A}\mathsf{S}) = \mathrm{tr}\mathbf{A}\boldsymbol{\Sigma}, \quad \mathrm{Var}(\mathrm{tr}\mathbf{A}\mathsf{S}) = 2n^{-1}\mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma})^2. \tag{A3}$$

The following lemma is frequently used.

Lemma A.1. *Let* $\mathsf{W} = n\mathsf{S}$ *be a* $p \times p$ *random matrix which is distributed as a Wishart distribution* $\mathrm{W}_p(n, \boldsymbol{\Sigma})$*, and let* $\mathbf{A}$ *be a fixed* $p \times p$ *matrix. Under an asymptotic framework* A1*, it holds that*

$$\begin{aligned}\frac{\mathrm{tr}\mathbf{A}\mathsf{W} - \mathrm{E}(\mathrm{tr}\mathbf{A}\mathsf{W})}{\sqrt{\mathrm{Var}(\mathrm{tr}\mathbf{A}\mathsf{W})}} &= \frac{\mathrm{tr}\mathbf{A}\mathsf{S} - \mathrm{tr}\mathbf{A}\boldsymbol{\Sigma}}{\sqrt{\mathrm{Var}(\mathrm{tr}\mathbf{A}\mathsf{S})}} \\ &= \frac{\mathrm{tr}\mathbf{A}\mathsf{S} - \mathrm{tr}\mathbf{A}\boldsymbol{\Sigma}}{\sqrt{2n^{-1}\mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma})^2}} \xrightarrow{d} \mathrm{N}(0, 1).\end{aligned} \tag{A4}$$

Further, when $\lim \sqrt{n^{-1}\mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma})^2} = \gamma$*,*

$$\frac{\mathrm{tr}\mathbf{A}\mathsf{S} - \mathrm{tr}\mathbf{A}\boldsymbol{\Sigma}}{\lim\left\{\sqrt{\mathrm{Var}(\mathrm{tr}\mathbf{A}\mathsf{S})}\right\}} = \frac{\mathrm{tr}\mathbf{A}\mathsf{S} - \mathrm{tr}\mathbf{A}\boldsymbol{\Sigma}}{\sqrt{2}\gamma} \xrightarrow{d} \mathrm{N}(0, 1). \tag{A5}$$

Proof. We may write

$$\begin{aligned}\mathrm{tr}\mathbf{A}\mathsf{W} &= \mathrm{tr}\boldsymbol{\Sigma}^{1/2}\mathbf{A}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2}\mathsf{W}\boldsymbol{\Sigma}^{-1/2}, \\ &= \mathrm{tr}\tilde{\mathbf{A}}\mathsf{X}\mathsf{X}',\end{aligned}$$

where

$$\tilde{\mathbf{A}} = \mathbf{\Sigma}^{1/2}\mathbf{A}\mathbf{\Sigma}^{1/2}, \quad \mathbf{X} \sim \mathrm{N}_{p\times n}(\mathbf{O}, \mathbf{I}_p \otimes \mathbf{I}_n)$$

Let $\mathbf{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$, then $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are independently and identically distributed as $\mathrm{N}_p(\mathbf{0}, \mathbf{I}_p)$ and

$$\mathrm{tr}\mathbf{A}\mathbf{W} = \mathrm{tr}\tilde{\mathbf{A}}\mathbf{X}\mathbf{X}' = \sum_{i=1}^{n} \boldsymbol{x}_i'\tilde{\mathbf{A}}\boldsymbol{x}_i.$$

It is easely seen that

$$\begin{aligned} &\mathrm{E}[\boldsymbol{x}_i'\tilde{\mathbf{A}}\boldsymbol{x}_i] = \mathrm{tr}\tilde{\mathbf{A}} = \mathrm{tr}\mathbf{A}\mathbf{\Sigma}, \quad \mathrm{Var}(\boldsymbol{x}_i'\tilde{\mathbf{A}}\boldsymbol{x}_i) = 2(\mathrm{tr}\tilde{\mathbf{A}}^2) = 2\mathrm{tr}(\mathbf{A}\mathbf{\Sigma})^2, \\ &\mathrm{E}[\mathrm{tr}\mathbf{A}\mathbf{W}] = n\mathrm{tr}\mathbf{A}\mathbf{\Sigma}, \quad \mathrm{Var}(\mathrm{tr}\mathbf{A}\mathbf{W}) = 2n\mathrm{tr}(\mathbf{A}\mathbf{\Sigma})^2. \end{aligned}$$

Let

$$Z_i = \frac{\boldsymbol{x}_i'\tilde{\mathbf{A}}\boldsymbol{x}_i - \mathrm{E}[\boldsymbol{x}_i'\tilde{\mathbf{A}}\boldsymbol{x}_i]}{\sqrt{\mathrm{Var}(\boldsymbol{x}_i'\tilde{\mathbf{A}}\boldsymbol{x}_i)}}, \quad i = 1, 2, \ldots, n.$$

Noting that $\mathrm{E}(Z_i) = 0$, $\mathrm{Var}(Z_i) = 1$ and $Z_1, Z_2, \ldots, Z_n$ are independent, we have

$$\sqrt{n}\bar{Z} = (Z_1 + \cdots + Z_n)/\sqrt{n} \xrightarrow{d} \mathrm{N}(0, 1).$$

We have

$$\begin{aligned} \sqrt{n}\bar{Z} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{\boldsymbol{x}_i'\tilde{\mathbf{A}}\boldsymbol{x}_i - \mathrm{tr}\tilde{\mathbf{A}}}{\sqrt{2(\mathrm{tr}\tilde{\mathbf{A}}^2)}} \\ &= \frac{1}{\sqrt{2n\mathrm{tr}(\mathbf{A}\mathbf{\Sigma})^2}}(\mathrm{tr}\mathbf{A}\mathbf{W} - n\mathrm{tr}\mathbf{A}\mathbf{\Sigma}) = \frac{\mathrm{tr}\mathbf{A}\mathbf{W} - \mathrm{E}[\mathrm{tr}\mathbf{A}\mathbf{W}]}{\mathrm{Var}(\mathrm{tr}\mathbf{A}\mathbf{W})}. \end{aligned}$$

The result (A5) is obtained from (A4). This completes the proof.

A2. The Limiting Distributions of $(n/p)a_0$, $(n/p)a_1$ and $(n/p)a_2$

Consider the quantities

$$\frac{n}{p}a_i = \frac{n}{p}\mathrm{tr}\mathbf{C}_i\mathbf{S}, \quad i = 0, 1, 2. \tag{A6}$$

where $n\mathbf{S} \sim \mathrm{W}_p(n, \boldsymbol{\Sigma})$. Under an asymptotic framework A1 and $\boldsymbol{\Sigma} = \sigma^2(\rho^{|i-j|})$, from Lemma A.1 it follows that $\frac{n}{p}a_i, i = 0, 1, 2$ have asymptotically normal. In the following we give their means and variances.

The mean and variance of $(n/p)a_0$

$$\begin{aligned}
\mathrm{E}\left[\frac{n}{p}a_0\right] &= \mathrm{E}\left[\frac{1}{p}\mathrm{tr}(n\mathbf{S})\right] = \frac{1}{p}n\mathrm{tr}\boldsymbol{\Sigma} = \frac{n}{p}p\sigma^2 = n\sigma^2, \\
\mathrm{Var}\left(\frac{n}{p}a_0\right) &= \frac{1}{p^2}\mathrm{Var}\left(\mathrm{tr}n\mathbf{S}\right) = \frac{1}{p^2}2n\mathrm{tr}\boldsymbol{\Sigma}^2 \\
&= \frac{\sigma^4}{p^2}2n\left\{p + 2\sum_{i=1}^{p-1}(p-i)\rho^{2i}\right\} \\
&= \frac{\sigma^4}{p^2}2n\left\{p + \frac{2\rho^2}{(1-\rho^2)^2}\left\{p(1-\rho^2) - 1 + \rho^{2p}\right\}\right\} \\
&\to \frac{2}{c}\left\{1 + \frac{2\rho^2}{(1-\rho^2)}\right\}\sigma^4 = \frac{2(1+\rho^2)}{c(1-\rho^2)}\sigma^4.
\end{aligned}$$

The mean and variance of $(n/p)a_1$

$$\begin{aligned}
\mathrm{E}\left[\frac{n}{p}a_1\right] &= \mathrm{E}\left[\frac{1}{p}\mathrm{tr}n\mathbf{C}_1\mathbf{S}\right] = \frac{1}{p}n\mathrm{tr}\mathbf{C}_1\boldsymbol{\Sigma} = \frac{n}{p}(p-2)\sigma^2, \\
\mathrm{Var}\left(\frac{n}{p}a_1\right) &= \frac{1}{p^2}\mathrm{Var}\left(\mathrm{tr}n\mathbf{C}_1\mathbf{S}\right) = \frac{1}{p^2}2n\mathrm{tr}(\mathbf{C}_1\boldsymbol{\Sigma})^2 \\
&= \frac{\sigma^4}{p^2}2n\left\{p - 2 + 2\sum_{i=1}^{p-3}(p-2-i)\rho^{2i}\right\} \\
&= \frac{\sigma^4}{p^2}2n\left\{p - 2 + \frac{2\rho^2}{(1-\rho^2)^2}\left\{(p-2)(1-\rho^2) - 1 + \rho^{2(p-2)}\right\}\right\} \\
&\to \frac{2}{c}\left\{1 + \frac{2\rho^2}{(1-\rho^2)}\right\}\sigma^4 = \frac{2(1+\rho^2)}{c(1-\rho^2)}\sigma^4.
\end{aligned}$$

The mean and variance of $(n/p)a_2$

$$\begin{aligned}
\mathrm{E}\left[\frac{n}{p}a_2\right] &= \mathrm{E}\left[\frac{1}{p}\mathrm{tr}n\mathbf{C}_2\mathbf{S}\right] = \frac{1}{p}n\mathrm{tr}\mathbf{C}_2\mathbf{\Sigma} = \frac{n}{p}(p-1)\sigma^2\rho, \\
\mathrm{Var}\left(\frac{n}{p}a_2\right) &= \frac{1}{p^2}\mathrm{Var}\left(\mathrm{tr}n\mathbf{C}_2\mathbf{S}\right) = \frac{1}{p^2}2n\mathrm{tr}(\mathbf{C}_2\mathbf{\Sigma})^2 \\
&= \frac{\sigma^4}{p^2}n\left\{p-1+(5p-9)\rho^2+4\sum_{i=1}^{p-3}(p-2-i)\rho^{2(i+1)}\right\} \\
&= \frac{\sigma^4}{p^2}n\left\{p-1+(5p-9)\rho^2+\frac{4\rho^4}{(1-\rho^2)^2}\left\{(p-2)(1-\rho^2)-1+\rho^{2p}\right\}\right\} \\
&\to \frac{\sigma^4}{c}\left\{1+5\rho^2+\frac{4\rho^4}{(1-\rho^2)}\right\} = \frac{1+4\rho^2-\rho^4}{c(1-\rho^2)}\sigma^4.
\end{aligned}$$

A3. The proof of Theorem 3.1(1)

For deriving asymptotic distributions of $\hat{\rho}$ and $\hat{\sigma}^2$, we use the following delta method. Consider a k-variate function $\boldsymbol{g}(\boldsymbol{x}) = (g_1(\boldsymbol{x}), \dots, g_k(\boldsymbol{x}))$, where $\boldsymbol{x} = (x_1, \dots, x_m)'$. Suppose that an $m-$variate random variate $\boldsymbol{X}$ is distributed as

$$\sqrt{n}(\boldsymbol{X}-\boldsymbol{\theta}) \xrightarrow{d} \mathrm{N}_m(\mathbf{0}, \mathbf{\Gamma}).$$

Then, it is known (see, e.g., Anderson (2003, p.132)) that

$$\sqrt{n}(\boldsymbol{g}(\boldsymbol{X})-\boldsymbol{g}(\boldsymbol{\theta})) \xrightarrow{d} \mathrm{N}_k(\mathbf{0}, \mathbf{H}\mathbf{\Gamma}\mathbf{H}'),$$

where

$$\mathbf{H} = \left(\left.\frac{\partial g_i(\boldsymbol{x})}{\partial x_j}\right|_{\boldsymbol{x}=\boldsymbol{\theta}}\right).$$

Let $\boldsymbol{X} = (X_1, X_2)'$, where $X_1 = p^{-1}a_1, X_2 = p^{-1}a_2$. Then,

$$n(\boldsymbol{X}-\boldsymbol{\theta}) \xrightarrow{d} \mathrm{N}_2(\mathbf{0}, \mathbf{\Gamma}),$$

where

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad \theta_1 = \frac{p-2}{p}\sigma^2, \quad \theta_2 = \frac{p-1}{p}\sigma^2\rho,$$
$$\mathbf{\Gamma} = \frac{\sigma^4}{c(1-\rho^2)}\begin{pmatrix} 2(1+\rho^2) & 4\rho \\ 4\rho & 1+4\rho^2-\rho^4 \end{pmatrix}.$$

Noting that $\hat{\rho} = \frac{a_2}{a_1} = \frac{p^{-1}a_2}{p^{-1}a_1}$, we define

$$g(x_1, x_2) = \frac{x_2}{x_1}.$$

Then

$$g(\boldsymbol{\theta}) = g\left(\frac{p-2}{p}\sigma^2, \frac{p-1}{p}\sigma^2\rho\right) = \frac{p-1}{p-2}\rho.$$

We have

$$\begin{aligned}
\boldsymbol{h}' &= (g_{x_1}, g_{x_2})|_{\boldsymbol{x}=\boldsymbol{\theta}} = \left(-\frac{x_2}{x_1^2}, \frac{1}{x_1}\right)\bigg|_{\boldsymbol{x}=\boldsymbol{\theta}} \\
&= \left(-\frac{\frac{p-1}{p}\sigma^2\rho}{\frac{(p-2)^2}{p^2}\sigma^4}, \frac{p}{(p-2)\sigma^2}\right) \to \left(-\frac{\rho}{\sigma^2}, \frac{1}{\sigma^2}\right) \\
\boldsymbol{h}'\boldsymbol{\Gamma}\boldsymbol{h} &= \frac{\sigma^4}{c(1-\rho^2)}\left(-\frac{\rho}{\sigma^2}, \frac{1}{\sigma^2}\right)\begin{pmatrix} 2(1+\rho^2) & 4\rho \\ 4\rho & 1+4\rho^2-\rho^4 \end{pmatrix}\left(-\frac{\rho}{\sigma^2}, \frac{1}{\sigma^2}\right)' \\
&= \frac{1}{c}(1-\rho^2).
\end{aligned}$$

This implies

$$n\left(\hat{\rho} - \frac{p-1}{p-2}\rho\right) \xrightarrow{d} \mathrm{N}\left(0, \frac{1}{c}(1-\rho^2)\right).$$

A4. The Limiting Distribution of $(s_{11}, s_{pp}, a_1, a_2)$

The statistics s_{11} and s_{pp} are expressed as

$$s_{11} = \mathrm{tr}\mathbf{A}_{11}\mathbf{S}, \quad \text{and} \quad s_{11} = \mathrm{tr}\mathbf{A}_{pp}\mathbf{S},$$

respectively, where

$$\mathbf{A}_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \mathbf{A}_{pp} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Therefore, any linear combination of $(s_{11}, s_{pp}, a_1, a_2)$ can be written as

$$k_1 s_{11} + k_2 s_{pp} + k_3 a_1 + k_4 a_2 = \mathrm{tr}\mathbf{A}\mathbf{S},$$

where $\mathbf{A} = \mathrm{tr}(\mathbf{A}_{11} + \mathbf{A}_{pp} + \mathbf{C}_1 + \mathbf{C}_2)$. So, $(s_{11}, s_{pp}, a_1, a_2)$ is asymptotically normal. The means, variance and covariances on s_{11} and s_{pp} are given as follows.

$$
\begin{aligned}
\mathrm{E}[ns_{11}] &= n\sigma^2, \\
\mathrm{Var}\,(ns_{11}) &= 2n\sigma^4, \\
\mathrm{Cov}\left(ns_{11}, \frac{n}{p}a_1\right) &= \frac{1}{p}2n\sigma^4 \sum_{i=1}^{p-2} \rho^{2i} = 2\frac{n}{p}\sigma^4 \frac{\rho^2 - \rho^{2(p-1)}}{1-\rho^2} \to \frac{2}{c}\sigma^4 \frac{\rho^2}{1-\rho^2}, \\
\mathrm{Cov}\left(ns_{11}, \frac{n}{p}a_2\right) &= \frac{1}{p}2n\sigma^4 \sum_{i=1}^{p-2} \rho^{2i-1} = 2\frac{n}{p}\sigma^4 \frac{\rho - \rho^{2p-1}}{1-\rho^2} \to \frac{2}{c}\sigma^4 \frac{\rho}{1-\rho^2}, \\
\mathrm{E}[ns_{pp}] &= n\sigma^2, \\
\mathrm{Var}\,(ns_{pp}) &= 2n\sigma^4, \\
\mathrm{Cov}\left(ns_{pp}, \frac{n}{p}a_1\right) &= \frac{1}{p}2n\sigma^4 \sum_{i=1}^{p-2} \rho^{2i} = 2\frac{n}{p}\sigma^4 \frac{\rho^2 - \rho^{2(p-1)}}{1-\rho^2} \to \frac{2}{c}\sigma^4 \frac{\rho^2}{1-\rho^2}, \\
\mathrm{Cov}\left(ns_{pp}, \frac{n}{p}a_2\right) &= \frac{1}{p}2n\sigma^4 \sum_{i=1}^{p-2} \rho^{2i-1} = 2\frac{n}{p}\sigma^4 \frac{\rho - \rho^{2p-1}}{1-\rho^2} \to \frac{2}{c}\sigma^4 \frac{\rho}{1-\rho^2}, \\
\mathrm{Cov}(s_{11}, s_{pp}) &= 2n\sigma^4\rho^{2(p-1)}, \\
\mathrm{Cov}\left(\frac{n}{p}a_1, \frac{n}{p}a_2\right) &= \frac{1}{p^2}2n\mathrm{tr}\mathbf{C}_1\mathbf{\Sigma}\mathbf{C}_2\mathbf{\Sigma} \\
&= \frac{1}{p^2}4n\sigma^4\frac{\rho}{(1-\rho^2)^2}\left\{(p-1)(1-\rho^2) - 1 + \rho^{2(p-1)}\right\} \to \frac{4}{c}\frac{\rho}{1-\rho^2}\sigma^4.
\end{aligned}
$$

We have the following distributional result:

$$
n\begin{pmatrix} \frac{1}{\sqrt{p}}s_{11} - \frac{1}{\sqrt{p}}\sigma^2 \\ \frac{1}{\sqrt{p}}s_{pp} - \frac{1}{\sqrt{p}}\sigma^2 \\ \frac{1}{p}a_1 - \frac{p-2}{p}\sigma^2 \\ \frac{1}{p}a_2 - \frac{p-1}{p}\sigma^2\rho \end{pmatrix} \xrightarrow{d} \mathrm{N}_4\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{\Gamma}\right),
$$

where the covariance matrix $\mathbf{\Gamma}$ is given by

$$
\frac{\sigma^4}{c(1-\rho^2)}\begin{pmatrix} 2(1-\rho^2) & (1-\rho^2)\rho^{2(p-1)} & 2\rho^2 & 2\rho^2 \\ (1-\rho^2)\rho^{2(p-1)} & 2(1-\rho^2) & 2\rho^2 & 2\rho^2 \\ 2\rho^2 & 2\rho^2 & 2(1+\rho^2) & 4\rho \\ 2\rho^2 & 2\rho^2 & 4\rho & 1+4\rho^2-\rho^4. \end{pmatrix}
$$

A5. The Proof of Theorem 3.2(2)

Noting that $a_0 = a_1 + s_{11} + s_{pp}$, we can express $\widehat{\sigma}^2$ as

$$
\begin{aligned}
\widehat{\sigma}^2 &= \frac{n}{N}\frac{1}{p(1-\hat{\rho}^2)}(a_1\hat{\rho}^2 - 2a_2\hat{\rho} + a_0) \\
&= \frac{n}{N}\frac{a_1^2}{p(a_1^2 - a_2^2)}\left(a_1\frac{a_2^2}{a_1^2} - 2a_2\frac{a_2}{a_1} + a_0\right) \\
&= \frac{n}{N}\frac{1}{p(a_1^2 - a_2^2)}\left(-a_1a_2^2 + a_0a_1^2\right) \\
&= \frac{n}{N}\frac{1}{p(a_1^2 - a_2^2)}\left\{-a_1a_2^2 + (a_1 + s_{11} + s_{pp})a_1^2\right\} \\
&= \frac{n}{N}\left\{\frac{1}{p}a_1 + \frac{1}{p}(s_{11} + s_{pp})\frac{a_1^2}{a_1^2 - a_2^2}\right\} \\
&= \frac{n}{n-q}\left\{\frac{1}{p}a_1 + \frac{1}{\sqrt{p}}\left(\frac{1}{\sqrt{p}}s_{11} + \frac{1}{\sqrt{p}}s_{pp}\right)\frac{(a_1/p)^2}{(a_1/p)^2 - (a_2/p)^2}\right\} \\
&\approx \frac{1}{p}a_1 + \frac{1}{\sqrt{p}}\left(\frac{1}{\sqrt{p}}s_{11} + \frac{1}{\sqrt{p}}s_{pp}\right)\frac{(a_1/p)^2}{(a_1/p)^2 - (a_2/p)^2}.
\end{aligned}
$$

Let

$$(X_1, X_2, X_3, X_4) = ((1/\sqrt{p})s_{11}, (1/\sqrt{p})s_{22}, (1/p)a_1, (1/p)a_2).$$

Further, define $g(x_1, x_2, x_3, x_4)$ as

$$g(x_1, x_2, x_3, x_4) = x_3 + \frac{1}{\sqrt{p}}(x_1 + x_2)\frac{x_3^2}{x_3^2 - x_4^2}.$$

$$
\begin{aligned}
g(\boldsymbol{\theta}) &= g\left(\frac{1}{\sqrt{p}}\sigma^2, \frac{1}{\sqrt{p}}\sigma^2, \frac{p-2}{p}\sigma^2, \frac{p-1}{p}\sigma^2\right) \\
&= \frac{p-2}{p}\sigma^2 + \frac{1}{\sqrt{p}}\left(\frac{1}{\sqrt{p}}\sigma^2 + \frac{1}{\sqrt{p}}\sigma^2\right)\frac{(\frac{p-2}{p})^2}{(\frac{p-2}{p})^2 - (\frac{p-1}{p}\rho)^2} \\
&= \left\{\frac{p-2}{p} + \frac{2}{p}\frac{(p-2)^2}{(p-2)^2 - (p-1)^2\rho^2}\right\}\sigma^2.
\end{aligned}
$$

We have

$$g_{x_1}(\boldsymbol{\theta}) = \left(\frac{1}{\sqrt{p}}\frac{x_3^2}{x_3^2 - x_4^2}\right)\Bigg|_{\boldsymbol{x}=\boldsymbol{\theta}} = \frac{1}{\sqrt{p}}\frac{(\frac{p-2}{p})^2}{(\frac{p-2}{p})^2 - (\frac{p-1}{p}\rho)^2} \to 0,$$

$$g_{x_2}(\boldsymbol{\theta}) = \left(\frac{1}{\sqrt{p}}\frac{x_3^2}{x_3^2 - x_4^2}\right)\Bigg|_{\boldsymbol{x}=\boldsymbol{\theta}} = \frac{1}{\sqrt{p}}\frac{(\frac{p-2}{p})^2}{(\frac{p-2}{p})^2 - (\frac{p-1}{p}\rho)^2} \to 0,$$

$$\begin{aligned} g_{x_3}(\boldsymbol{\theta}) &= \left\{1 + \frac{1}{\sqrt{p}}(x_1 + x_2)\left(-2\frac{x_3 x_4^2}{(x_3^2 - x_4^2)^2}\right)\right\}\Bigg|_{\boldsymbol{x}=\boldsymbol{\theta}} \\ &= 1 - 2\frac{1}{\sqrt{p}}\left(\frac{1}{\sqrt{p}}\sigma^2 + \frac{1}{\sqrt{p}}\sigma^2\right)\left(\frac{(\frac{p-2}{p}\sigma^2)(\frac{p-1}{p}\sigma^2\rho)^2}{\{(\frac{p-2}{p}\sigma^2)^2 - (\frac{p-1}{p}\sigma^2\rho)^2\}^2}\right) \to 1, \end{aligned}$$

$$\begin{aligned} g_{x_4}(\boldsymbol{\theta}) &= \left\{\frac{1}{\sqrt{p}}(x_1 + x_2)\left(-2\frac{x_3^2 x_4}{(x_3^2 - x_4^2)^2}\right)\right\}\Bigg|_{\boldsymbol{x}=\boldsymbol{\theta}} \\ &= -2\frac{1}{\sqrt{p}}\left(\frac{1}{\sqrt{p}}\sigma^2 + \frac{1}{\sqrt{p}}\sigma^2\right)\left(\frac{(\frac{p-2}{p}\sigma^2)^2(\frac{p-1}{p}\sigma^2\rho)}{\{(\frac{p-2}{p}\sigma^2)^2 - (\frac{p-1}{p}\sigma^2\rho)^2\}^2}\right) \to 0. \end{aligned}$$

The $\mathbf{H}$ in our delta method is given by $\boldsymbol{h} = (0, 0, 1, 0)'$, and

$$\boldsymbol{h}'\boldsymbol{\Gamma}\boldsymbol{h} = 2\sigma^4\frac{(1+\rho^2)}{c(1-\rho^2)}$$

Therefor we have

$$n\left(\hat{\sigma}^2 - \left\{\frac{p-2}{p} + \frac{2}{p}\frac{(p-2)^2}{(p-2)^2 - (p-1)^2\rho^2}\right\}\sigma^2\right) \xrightarrow{d} \mathrm{N}\left(0, 2\sigma^4\frac{(1+\rho^2)}{c(1-\rho^2)}\right).$$

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