Tests for One and Two Mean Vectors and Simultaneous Confidence Intervals with Monotone Incomplete Data

[1]Ayaka Yagi* and Takashi Seo*

* *Department of Applied Mathematics*
Tokyo University of Science
1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

Abstract

In this study, we consider the problems of testing for a mean vector and testing the equality of two mean vectors with monotone missing data. We propose new test statistics similar to the simplified Hotelling's T^2-type test statistic for one-sample and two-sample problems under general-step monotone missing data. Approximate upper percentiles of this new statistics are provided by asymptotic expansion along with transformed test statistics based on Bartlett adjustment. Approximate simultaneous confidence intervals for pairwise comparisons among mean vectors are also presented. Furthermore, we investigated the asymptotic behavior of the proposed statistics using a Monte Carlo simulation, and the approximate accuracy of the proposed approximations and test statistics are provided and discussed. Finally, the numerical power of these statistics are given.

Key Words and Phrases: Asymptotic expansion; Bartlett correction; Hotelling's T^2-type test statistic; One-sample problem; Transformed test statistic; Two-sample problem; Pairwise comparison

1 Introduction

Statistical data analysis often includes missing values, and various studies have been conducted on statistical methods to address this issue. Missing data may or may not exhibit monotone or non-monotone patterns. In particular, for the monotone case, maximum likelihood estimators (MLE) of the mean vector and variance-covariance matrix are given in Anderson and Olkin (1985), Jinadasa and Tracy (1992), and Kanda and Fujikoshi (1998). For the problem of testing the mean vector in a one-sample case with monotone missing data, testing using a T^2-type test statistic have been discussed by Seko, Yamazaki, and Seo (2012), Romer and Richards (2013), and Kawasaki, Shutoh, and Seo

[1]yagi@rs.tus.ac.jp

(2018), among others. In contrast, testing using a simplified T^2-type test statistic have been considered in the context of general k-step monotone missing data by Krishnamoorthy and Pannala (1999) and Yagi, Seo, and Hanusz (2019). In particular, Yagi et al. (2019) provided an asymptotic expansion of the null distribution of the test statistic Q by decomposing $Q = Q_1 + Q_2$ and giving the asymptotic expansion of Q_1 and Q_2 for $k = 2$. They then used the results to derive an approximate upper percentile of Q and transformation statistics that improve the chi-square approximation. Here, Q_1 and Q_2 are asymptotically independent, but not exactly independent, and this is taken account to derive the exact null distribution of Q. However, for k-step monotone missing data ($k \geq 3$), due to the complexity, they derive the asymptotic expansion of the null distribution of $Q(= \sum_{i=1}^{k} Q_i)$ when $Q_i (i = 1, 2, \dots, k)$ are assumed to be independent of each other.

Therefore, in this study, we propose a new test statistic $Q_{\mathrm{M}}(= Q_1 + \sum_{i=1}^{k} R_i)$. Details are provided in the next section. This Q_{M} replaces $Q_i (i = 2, 3, \dots, k)$ in $Q(= \sum_{i=1}^{k} Q_i)$ by $R_i (= Q_i (1 + Q_{id})^{-1}) (i = 2, 3, \dots, k)$ which are independent of Q_1 and each other. We obtain its exact distribution and propose an approximate upper percentile and transformation statistics. The discussion applies equally to the two-sample case. In addition, we discuss approximate simultaneous confidence intervals based on this statistic for the two-sample case and pairwise comparisons between mean vectors. Throughout this paper, we assume that data are missing completely at random (MCAR). Note that Q_{M} in one- and two-sample cases for $k = 2$ has been discussed in Onozawa, Yagi, and Seo (2020), and we provide an extension to the general k-step in the present work. Pairwise comparisons between mean vectors in monotone missing data have been investigated in prior works such as those by Seko (2012), Yagi and Seo (2017) and so forth.

The remainder of this paper is organized as follows. In Section 2, an asymptotic expansion of the null distribution of a new test statistic similar to the simplified T^2 statistic is derived with three-step monotone missing data and k-step monotone missing data for the one-sample case. We also propose an approximate upper percentile for Q and transformed test statistics. In Section 3, we describe an extension of the expansion given in Section 2 to the two-sample case. In Section 4, the approximate simultaneous confidence intervals for multivariate pairwise comparisons among mean vectors when each dataset has k-step monotone missing observations are obtained. In Section 5, we describe the results of simulation studies on the upper percentiles of the proposed test statistics and empirical type I errors. We also compared the results with the upper percentiles of the statistic and other statistics given in previous studies. Numerical power for several statistics is discussed in Section 6. Finally, we conclude this paper in Section 7.

2 Test statistics for one-sample problem

2.1 Three-step case ($k = 3$)

We first consider the null distribution of a new test statistic similar to the simplified T^2 statistic for three-step monotone missing data.

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & \cdots & x_{1p_1} & x_{1,p_1+1} & \cdots & x_{1p_1+p_2} & x_{1,p_1+p_2+1} & \cdots & x_{1p} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{n_11} & \cdots & x_{n_1p_1} & x_{n_1,p_1+1} & \cdots & x_{n_1p_1+p_2} & x_{n_1,p_1+p_2+1} & \cdots & x_{n_1p} \\ x_{n_1+1,1} & \cdots & x_{n_1+1,p_1} & x_{n_1+1,p_1+1} & \cdots & x_{n_1+1,p_1+p_2} & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{n_1+n_2,1} & \cdots & x_{n_1+n_2,p_1} & x_{n_1+n_2,p_1+1} & \cdots & x_{n_1+n_2,p_1+p_2} & * & \cdots & * \\ x_{n_1+n_2+1,1} & \cdots & x_{n_1+n_2+1,p_1} & * & \cdots & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{n_1+n_2+n_3,1} & \cdots & x_{n_1+n_2+n_3,p_1} & * & \cdots & * & * & \cdots & * \end{pmatrix},$$

where "$*$" indicates a missing observation, $p = p_1 + p_2 + p_3$, and $n_1 > p$. Then, let

$$\boldsymbol{X} = \begin{pmatrix} \overbrace{\boldsymbol{X}_{11}}^{p_1} & \overbrace{\boldsymbol{X}_{12}}^{p_2} & \overbrace{\boldsymbol{X}_{13}}^{p_3} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & * \\ \boldsymbol{X}_{31} & * & * \end{pmatrix} \begin{matrix} \}n_1 = N_1 \\ \}n_2 \\ \}n_3 \end{matrix} \quad N_2, \quad N_3,$$

where rows of $\begin{pmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \boldsymbol{X}_{13} \end{pmatrix}$, $\begin{pmatrix} \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{pmatrix}$, and $\boldsymbol{X}_{31}$ are distributed as multivariate normal distributions. Further, we suppose that

$$\mathrm{vec}(\boldsymbol{X}'_{1(123)}) \sim N_{n_1p}(\mathrm{vec}(\boldsymbol{\mu}\boldsymbol{1}'_{n_1}), \boldsymbol{I}_{n_1} \otimes \boldsymbol{\Sigma}),$$
$$\mathrm{vec}(\boldsymbol{X}'_{2(12)}) \sim N_{n_2p_{(12)}}(\mathrm{vec}(\boldsymbol{\mu}_{(12)}\boldsymbol{1}'_{n_2}), \boldsymbol{I}_{n_2} \otimes \boldsymbol{\Sigma}_{(12)(12)}),$$
$$\mathrm{vec}(\boldsymbol{X}'_{31}) \sim N_{n_3p_1}(\mathrm{vec}(\boldsymbol{\mu}_1\boldsymbol{1}'_{n_3}), \boldsymbol{I}_{n_3} \otimes \boldsymbol{\Sigma}_{11}),$$

where $\boldsymbol{X}_{1(123)} = \begin{pmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \boldsymbol{X}_{13} \end{pmatrix}$, $\boldsymbol{X}_{2(12)} = \begin{pmatrix} \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{pmatrix}$, $p_{(12)} = p_1 + p_2$,

$$\mathbf{1}_{n_i} = \left.\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right\} n_i, \ i = 1, 2, 3, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \hline \boldsymbol{\mu}_3 \end{pmatrix} \begin{matrix} \}p_1 \\ \}p_2 \\ \}p_3 \end{matrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)} \\ \hline \boldsymbol{\mu}_3 \end{pmatrix},$$

and

$$\boldsymbol{\Sigma} = \left(\begin{array}{cc|c} \overbrace{\boldsymbol{\Sigma}_{11}}^{p_1} & \overbrace{\boldsymbol{\Sigma}_{12}}^{p_2} & \overbrace{\boldsymbol{\Sigma}_{13}}^{p_3} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \boldsymbol{\Sigma}_{23} \\ \hline \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33} \end{array} \right) \begin{matrix} \}p_1 \\ \}p_2 \\ \}p_3 \end{matrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\boldsymbol{\Sigma}_{(12)(12)}$}} & \boldsymbol{\Sigma}_{13} \\ & & \boldsymbol{\Sigma}_{23} \\ \hline \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33} \end{array} \right).$$

We define

$$\overline{\boldsymbol{x}}_{(123)1} = \frac{1}{N_3} \boldsymbol{X}'_{(123)1} \mathbf{1}_{N_3}, \quad \boldsymbol{S}_{(123)1} = \frac{1}{N_3 - 1} \left\{ \boldsymbol{X}'_{(123)1} \boldsymbol{X}_{(123)1} - N_3 \overline{\boldsymbol{x}}_{(123)1} \overline{\boldsymbol{x}}'_{(123)1} \right\},$$

$$\overline{\boldsymbol{x}}_{(12)(12)} = \frac{1}{N_2} \boldsymbol{X}'_{(12)(12)} \mathbf{1}_{N_2}, \quad \boldsymbol{S}_{(12)(12)} = \frac{1}{N_2 - 1} \left\{ \boldsymbol{X}'_{(12)(12)} \boldsymbol{X}_{(12)(12)} - N_2 \overline{\boldsymbol{x}}_{(12)(12)} \overline{\boldsymbol{x}}'_{(12)(12)} \right\},$$

$$\overline{\boldsymbol{x}}_{1(123)} = \frac{1}{N_1} \boldsymbol{X}'_{1(123)} \mathbf{1}_{N_1}, \quad \boldsymbol{S}_{1(123)} = \frac{1}{N_1 - 1} \left\{ \boldsymbol{X}'_{1(123)} \boldsymbol{X}_{1(123)} - N_1 \overline{\boldsymbol{x}}_{1(123)} \overline{\boldsymbol{x}}'_{1(123)} \right\},$$

where

$$\boldsymbol{X}_{(123)1} = \begin{pmatrix} \boldsymbol{X}_{11} \\ \boldsymbol{X}_{21} \\ \boldsymbol{X}_{31} \end{pmatrix}, \quad \boldsymbol{X}_{(12)(12)} = \begin{pmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{pmatrix},$$

and $N_i = \sum_{j=1}^{i} n_j, \ i = 1, 2, 3$. Further, we partition $\overline{\boldsymbol{x}}_{1(123)}$, $\boldsymbol{S}_{1(123)}$, $\overline{\boldsymbol{x}}_{(12)(12)}$, and $\boldsymbol{S}_{(12)(12)}$ as follows.

$$\overline{\boldsymbol{x}}_{1(123)} = \begin{pmatrix} \overline{\boldsymbol{x}}_{1(12)} \\ \overline{\boldsymbol{x}}_{13} \end{pmatrix} \begin{matrix} \}p_{(12)} \\ \}p_3 \end{matrix}, \quad \boldsymbol{S}_{1(123)} = \begin{pmatrix} \overbrace{\boldsymbol{S}_{1(123),11}}^{p_{(12)}} & \overbrace{\boldsymbol{S}_{1(123),12}}^{p_3} \\ \boldsymbol{S}_{1(123),21} & \boldsymbol{S}_{1(123),22} \end{pmatrix} \begin{matrix} \}p_{(12)} \\ \}p_3 \end{matrix},$$

$$\overline{\boldsymbol{x}}_{(12)(12)} = \begin{pmatrix} \overline{\boldsymbol{x}}_{(12)1} \\ \overline{\boldsymbol{x}}_{(12)2} \end{pmatrix} \begin{matrix} \}p_1 \\ \}p_2 \end{matrix}, \quad \boldsymbol{S}_{(12)(12)} = \begin{pmatrix} \overbrace{\boldsymbol{S}_{(12)(12),11}}^{p_1} & \overbrace{\boldsymbol{S}_{(12)(12),12}}^{p_2} \\ \boldsymbol{S}_{(12)(12),21} & \boldsymbol{S}_{(12)(12),22} \end{pmatrix} \begin{matrix} \}p_1 \\ \}p_2 \end{matrix}.$$

Consider

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 \text{ vs. } H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0, \tag{1}$$

where $\boldsymbol{\mu}_0$ is known. Without loss of generality, we can assume that $\boldsymbol{\mu}_0 = \mathbf{0}$. Then, we propose a test statistic

$$Q_{\mathrm{M}(1,3)} = Q_1 + R_2 + R_3, \tag{2}$$

where

$$
\begin{aligned}
Q_1 =& N_3\widehat{\boldsymbol{\eta}}_1'\widehat{\boldsymbol{\Delta}}_{11}^{-1}\widehat{\boldsymbol{\eta}}_1, \quad R_2 = \frac{Q_2}{1+Q_{2d}}, \quad R_3 = \frac{Q_3}{1+Q_{3d}},\\
Q_2 =& N_2\widehat{\boldsymbol{\eta}}_2'\widehat{\boldsymbol{\Delta}}_{22}^{-1}\widehat{\boldsymbol{\eta}}_2, \quad Q_3 = N_1\widehat{\boldsymbol{\eta}}_3'\widehat{\boldsymbol{\Delta}}_{33}^{-1}\widehat{\boldsymbol{\eta}}_3,\\
\widehat{\boldsymbol{\eta}}_1 =& \overline{\boldsymbol{x}}_{(123)1}, \quad \widehat{\boldsymbol{\Delta}}_{11} = \frac{1}{N_3}(N_3-1)\boldsymbol{S}_{(123)1},\\
\widehat{\boldsymbol{\eta}}_2 =& \overline{\boldsymbol{x}}_{(12)2} - \boldsymbol{S}_{(12)(12),21}\boldsymbol{S}_{(12)(12),11}^{-1}\overline{\boldsymbol{x}}_{(12)1},\\
\widehat{\boldsymbol{\Delta}}_{22} =& \frac{N_2-1}{N_2}\Big[\boldsymbol{S}_{(12)(12),22} - \boldsymbol{S}_{(12)(12),21}\boldsymbol{S}_{(12)(12),11}^{-1}\boldsymbol{S}_{(12)(12),12}\Big],\\
\widehat{\boldsymbol{\eta}}_3 =& \overline{\boldsymbol{x}}_{13} - \boldsymbol{S}_{1(123),21}\boldsymbol{S}_{1(123),11}^{-1}\overline{\boldsymbol{x}}_{1(12)},
\end{aligned}
$$

and

$$
\widehat{\boldsymbol{\Delta}}_{33} = \frac{N_1-1}{N_1}\Big[\boldsymbol{S}_{1(123),22} - \boldsymbol{S}_{1(123),21}\boldsymbol{S}_{1(123),11}^{-1}\boldsymbol{S}_{1(123),12}\Big].
$$

To test H_0, we derive the null distribution of $Q_{M(1,3)}$. Because Q_1, R_2, and R_3 are mutually independent,

$$
\mathrm{E}[\exp(itQ_{\mathrm{M}(1,3)})] = \mathrm{E}[\exp(itQ_1)]\mathrm{E}[\exp(itR_2)]\mathrm{E}[\exp(itR_3)].
$$

Hence, $\mathrm{E}[\exp(itQ_1)]$, $\mathrm{E}[\exp(itR_2)]$, and $\mathrm{E}[\exp(itR_3)]$ are needed. For large N_3, from the result of Yagi et al. (2019), a stochastic expansion of Q_1 in (2) is given by

$$
\mathrm{E}[\exp(itQ_1)] = u^{-\frac{1}{2}p_1} + \frac{1}{N_3}\sum_{j=0}^{2}\beta_{j,1}u^{-\frac{1}{2}p_1-j} + O(N_3^{-2}),
$$

where

$$
\beta_{0,1} = -\frac{1}{4}p_1(p_1+2), \quad \beta_{1,1} = 0, \quad \beta_{2,1} = -\beta_{0,1}, \quad u = 1-2it.
$$

Then, we consider the stochastic expansion of R_2 and R_3 in (2). As a reult, we obtain

$$
\mathrm{E}[\exp(itR_2)] = u^{-\frac{1}{2}p_2} + \frac{1}{N_2}\sum_{j=0}^{2}\gamma_{j,2}u^{-\frac{1}{2}p_2-j} + O(N_2^{-2}),
$$

$$
\mathrm{E}[\exp(itR_3)] = u^{-\frac{1}{2}p_3} + \frac{1}{N_1}\sum_{j=0}^{2}\gamma_{j,3}u^{-\frac{1}{2}p_3-j} + O(N_1^{-2}),
$$

where

$$
\begin{aligned}
\gamma_{0,2} &= -\frac{1}{4}p_2(2p_1+p_2+2), \quad \gamma_{1,2} = \frac{1}{2}p_1p_2, \quad \gamma_{2,2} = \frac{1}{4}p_2(p_2+2),\\
\gamma_{0,3} &= -\frac{1}{4}p_3(2p_{(12)}+p_3+2), \quad \gamma_{1,3} = \frac{1}{2}p_{(12)}p_3, \quad \gamma_{2,3} = \frac{1}{4}p_3(p_3+2).
\end{aligned}
$$

For the derivations of the distribution function of R_2 and R_3, see the Appendix. Based on this background, we have the following theorem.

Theorem 1

For large n_1, the characteristic function of $Q_{\mathrm{M}(1,3)}$ can be expanded as

$$\mathrm{E}[\exp(itQ_{\mathrm{M}(1,3)})] = (1-2it)^{-\frac{1}{2}p} + \frac{1}{n_1}\sum_{j=0}^{2}\gamma_j(1-2it)^{-\frac{1}{2}p-j} + O(n_1^{-2}),$$

where

$$\begin{aligned}
\gamma_0 &= -\frac{1}{4}\left\{\frac{1}{1+r_2+r_3}p_1(p_1+2) + \frac{1}{1+r_2}p_2(2p_1+p_2+2) + p_3(2p_{(12)}+p_3+2)\right\}, \\
\gamma_1 &= \frac{1}{2}\left\{\frac{1}{1+r_2}p_1p_2 + p_{(12)}p_3)\right\}, \\
\gamma_2 &= \frac{1}{4}\left\{\frac{1}{1+r_2+r_3}p_1(p_1+2) + \frac{1}{1+r_2}p_2(p_2+2) + p_3(p_3+2)\right\}, \quad r_2 = \frac{n_2}{n_1}, \quad r_3 = \frac{n_3}{n_1},
\end{aligned}$$

and r_i is a positive constant. Also, the distribution of $Q_{\mathrm{M}(1,3)}$ is

$$\Pr(Q_{\mathrm{M}(1,3)} \le x) = G_p(x) + \frac{1}{n_1}\sum_{j=0}^{2}\gamma_j G_{p+2j}(x) + O(n_1^{-2}),$$

where $G_f(x)$ is the distribution function of a chi-squared variate with f degrees of freedom.

From the result of Theorem 1, its upper 100α percentiles can be expanded as

$$q_{\mathrm{M}(1,3)}(\alpha) = \chi_p^2(\alpha) - \frac{1}{n_1}\left[\frac{2\chi_p^2(\alpha)}{p}\left\{\gamma_0 - \frac{\gamma_2}{p+2}\chi_p^2(\alpha)\right\}\right] + O(n_1^{-2}),$$

where $\chi_p^2(\alpha)$ is the upper 100α percentile of chi-squared distribution with p degrees of freedom. Therefore, an approximation to the upper 100α percentile of $Q_{\mathrm{M}(1,3)}$ is proposed as follows:

$$q_{\mathrm{M}(1,3)\mathrm{AE}}(\alpha) = \chi_p^2(\alpha) - \frac{1}{n_1}\frac{2\chi_p^2(\alpha)}{p}\left\{\gamma_0 - \frac{\gamma_2}{p+2}\chi_p^2(\alpha)\right\}.$$

2.2 General-step case

Next consider the case of k-step monotone missing data ($k \geq 2$):

$$
\boldsymbol{X} = \begin{pmatrix}
\overbrace{\boldsymbol{X}_{11}}^{p_1} & \overbrace{\boldsymbol{X}_{12}}^{p_2} & \cdots & \overbrace{\boldsymbol{X}_{1,k-1}}^{p_{k-1}} & \overbrace{\boldsymbol{X}_{1k}}^{p_k} \\
\boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \cdots & \boldsymbol{X}_{2,k-1} & * \\
\vdots & \vdots & & & \vdots \\
\boldsymbol{X}_{k-1,1} & \boldsymbol{X}_{k-1,2} & * & \cdots & * \\
\boldsymbol{X}_{k1} & * & \cdots & \cdots & *
\end{pmatrix}
\begin{matrix} \}n_1 \\ \}n_2 \\ \\ \}n_{k-1} \\ \}n_k \end{matrix}, \quad \ell = 1, 2,
$$

where $\boldsymbol{X}_{ij}$ is a $n_i \times p_j$ block matrix ($i = 1, 2, \ldots, k; j = 1, 2, \ldots, k - i + 1$), and "$*$" indicates a missing part. For the ith step data ($i = 1, 2, \ldots, k$), let

$$
\boldsymbol{X}_{i(12\ldots,k-i+1)} = (\boldsymbol{X}_{i1} \;\; \boldsymbol{X}_{i2} \;\; \cdots \;\; \boldsymbol{X}_{i,k-i+1}).
$$

Then, we assume that the rows of $\boldsymbol{X}_{i(12\ldots,k-i+1)}$ are mutually independent and

$$
\mathrm{vec}(\boldsymbol{X}'_{i(12\ldots,k-i+1)}) \sim N_{n_i p_{(12\ldots,k-i+1)}}(\mathrm{vec}(\boldsymbol{\mu}_{(12\ldots,k-i+1)}\boldsymbol{1}'_{n_i}), \boldsymbol{I}_{n_i} \otimes \boldsymbol{\Sigma}_{(12\ldots,k-i+1)(12\ldots,k-i+1)}),
$$
$$
i = 1, 2, \ldots, k,
$$

where $p_{(12\ldots,k-i+1)} = \sum_{j=1}^{k-i+1} p_j$,

$$
\underset{p_{(12\ldots,k-i+1)} \times 1}{\boldsymbol{\mu}_{(12\ldots,k-i+1)}} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \vdots \\ \boldsymbol{\mu}_{k-i+1} \end{pmatrix} \begin{matrix} \}p_1 \\ \}p_2 \\ \\ \}p_{k-i+1} \end{matrix},
$$

$$
\underset{p_{(12,\ldots,k-i+1)} \times p_{(12,\ldots,k-i+1)}}{\boldsymbol{\Sigma}_{(12,\ldots,k-i+1)(12,\ldots,k-i+1)}} = \begin{pmatrix}
\overbrace{\boldsymbol{\Sigma}_{11}}^{p_1} & \overbrace{\boldsymbol{\Sigma}_{12}}^{p_2} & \cdots & \overbrace{\boldsymbol{\Sigma}_{1,k-i+1}}^{p_{k-i+1}} \\
\boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \cdots & \boldsymbol{\Sigma}_{2,k-i+1} \\
\vdots & & & \vdots \\
\boldsymbol{\Sigma}_{k-i+1,1} & \boldsymbol{\Sigma}_{k-i+1,2} & \cdots & \boldsymbol{\Sigma}_{k-i+1,k-i+1}
\end{pmatrix} \begin{matrix} \}p_1 \\ \}p_2 \\ \\ \}p_{k-i+1} \end{matrix}.
$$

Further, let

$$
\boldsymbol{X}_{(12\ldots,k-i+1)(12\ldots i)} = \begin{pmatrix} \boldsymbol{X}_{11} & \cdots & \boldsymbol{X}_{1i} \\ \vdots & & \vdots \\ \boldsymbol{X}_{k-i+1,1} & \cdots & \boldsymbol{X}_{k-i+1,i} \end{pmatrix}, \quad i = 1, 2, \ldots, k,
$$

where $\boldsymbol{X}_{(12\ldots,k-i+1)(12\ldots i)}$ is a $N_{k-i+1}(=\sum_{j=1}^{k-i+1} n_j) \times p_{(12\ldots i)}$ matrix, and let

$$\begin{aligned}
\overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots i)} =& \frac{1}{N_{k-i+1}} \boldsymbol{X}_{(12\ldots,k-i+1)(12\ldots i)} \mathbf{1}_{N_{k-i+1}}, \\
\boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i)} =& \frac{1}{N_{k-i+1}-1} \Big\{ \boldsymbol{X}'_{(12\ldots,k-i+1)(12\ldots i)} \boldsymbol{X}_{(12\ldots,k-i+1)(12\ldots i)} \\
& - N_{k-i+1} \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots i)} \overline{\boldsymbol{x}}'_{(12\ldots,k-i+1)(12\ldots i)} \Big\}.
\end{aligned}$$

Further, for $i = 2, 3, \ldots, k$, we partition

$$\overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots i)} = \begin{pmatrix} \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots,i-1)} \\ \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)i} \end{pmatrix} \begin{matrix} \}p_{(12\ldots,i-1)} \\ \}p_i \end{matrix},$$

$$\boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i)} = \begin{pmatrix} \overbrace{\boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i),11}}^{p_{(12\ldots,i-1)}} & \overbrace{\boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i),12}}^{p_i} \\ \boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i),21} & \boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i),22} \end{pmatrix} \begin{matrix} \}p_{(12\ldots,i-1)} \\ \}p_i \end{matrix}.$$

Yagi et al. (2019) gave the simplified T^2 statistic

$$Q_{(1,k)} = \sum_{i=1}^{k} Q_i, \tag{3}$$

whereas we propose a test statistic for the hypothesis (1) as follows.

$$Q_{\mathrm{M}(1,k)} = Q_1 + \sum_{i=2}^{k} R_i \tag{4}$$

where

$$\begin{aligned}
Q_i =& N_{k-i+1} \widehat{\boldsymbol{\eta}}'_i \widehat{\boldsymbol{\Delta}}_{ii}^{-1} \widehat{\boldsymbol{\eta}}_i, \quad i = 1, 2, \ldots, k, \quad R_i = \frac{Q_i}{1 + Q_{id}}, \\
Q_{id} =& \frac{N_{k-i+1}}{N_{k-i+1}-1} \overline{\boldsymbol{x}}'_{(12\ldots,k-i+1)(12\ldots,i-1)} \boldsymbol{S}^{-1}_{(12\ldots,k-i+1)(12\ldots i),11} \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots,i-1)}, \ i = 2, 3, \ldots, k, \\
\widehat{\boldsymbol{\eta}}_1 =& \overline{\boldsymbol{x}}_{(12\ldots k)1}, \quad \widehat{\boldsymbol{\Delta}}_{11} = \frac{1}{N_k}(N_k - 1)\boldsymbol{S}_{(12\ldots k)1}, \\
\widehat{\boldsymbol{\eta}}_i =& \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)i} - \boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i),21} \boldsymbol{S}^{-1}_{(12\ldots,k-i+1)(12\ldots i),11} \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots,i-1)}, \\
\widehat{\boldsymbol{\Delta}}_{ii} =& \frac{N_{k-i+1}-1}{N_{k-i+1}} \Big[\boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i),22} \\
& - \boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i),21} \boldsymbol{S}^{-1}_{(12\ldots,k-i+1)(12\ldots i),11} \boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i),12} \Big], \quad i = 2, 3, \ldots, k.
\end{aligned}$$

The following theorem provides the characteristic function of $Q_{\mathrm{M}(1,k)}$.

Theorem 2

For large n_1, the characteristic function of $Q_{\mathrm{M}(1,k)}$ can be expanded as

$$\mathrm{E}[\exp(itQ_{\mathrm{M}(1,k)})] = (1-2it)^{-\frac{1}{2}p} + \frac{1}{n_1}\sum_{j=0}^{2}\gamma_j(1-2it)^{-\frac{1}{2}p-j} + O(n_1^{-2}),$$

where $p = \sum_{i=1}^{k} p_i$,

$$\gamma_j = \frac{1}{(\sum_{j=1}^{k} r_j)}\beta_{j,1} + \sum_{i=2}^{k}\frac{1}{(\sum_{j=1}^{k-i+1} r_j)}\gamma_{j,i}, \quad j=0,1,2, \quad r_i = \frac{n_i}{n_1}, \quad i=1,2,\ldots,k,$$

$$\beta_{0,1} = -\frac{1}{4}p_1(p_1+2), \quad \beta_{1,1} = 0, \quad \beta_{2,1} = -\beta_{0,1},$$

$$\gamma_{0,i} = -\frac{1}{4}p_i(2p_{(12\ldots,i-1)} + p_i + 2), \quad \gamma_{1,i} = \frac{1}{2}p_{(12\ldots,i-1)}p_i, \quad \gamma_{2,i} = \frac{1}{4}p_i(p_i+2), \quad i=2,3,\ldots,k,$$

and r_i is a positive constant. Moreover, the distribution function of $Q_{\mathrm{M}(1,k)}$ is

$$\Pr(Q_{\mathrm{M}(1,k)} \le x) = G_p(x) + \frac{1}{n_1}\sum_{j=0}^{2}\gamma_j G_{p+2j}(x) + O(n_1^{-2}),$$

where $G_f(x)$ is the distribution function of a chi-squared variate with f degrees of freedom.

For the derivations of Theorem 2, see the Appendix. Therefore, its upper 100α percentiles can be expanded as

$$q_{\mathrm{M}(1,k)}(\alpha) = \chi_p^2(\alpha) - \frac{1}{n_1}\left[\frac{2\chi_p^2(\alpha)}{p}\left\{\gamma_0 - \frac{\gamma_2}{p+2}\chi_p^2(\alpha)\right\}\right] + O(n_1^{-2}),$$

where $\chi_p^2(\alpha)$ is the upper 100α percentile of chi-squared distribution with p degrees of freedom. From the result of Theorem 2, an approximation to the upper 100α percentile of $Q_{\mathrm{M}(1,k)}$ is given by

$$q_{\mathrm{M}(1,k)\mathrm{AE}}(\alpha) = \chi_p^2(\alpha) - \frac{1}{n_1}\frac{2\chi_p^2(\alpha)}{p}\left\{\gamma_0 - \frac{\gamma_2}{p+2}\chi_p^2(\alpha)\right\}. \quad (5)$$

As another approximate upper 100α percentile of $Q_{\mathrm{M}(1,k)}$, we propose

$$q_{\mathrm{M}(1,k)\mathrm{KP}}(\alpha) = d_{(1,k)}F_{p,\nu_{(1,k)}}(\alpha) \quad (\text{for } n_1 > p+4), \quad (6)$$

where $F_{p,\nu_{(1,k)}}(\alpha)$ is the upper 100α percentile of F-distribution with p and $\nu_{(1,k)}$ degrees of freedom,

$$
\begin{aligned}
d_{(1,k)} &= G_{1(1,k)} \frac{\nu_{(1,k)} - 2}{\nu_{(1,k)}}, \quad \nu_{(1,k)} = \frac{4pG_{2(1,k)} - 2(p+2)G_{1(1,k)}^2}{pG_{2(1,k)} - (p+2)G_{1(1,k)}^2},\\
G_{1(1,k)} &= \mathrm{E}[Q_{\mathrm{M}(1,k)}] = \mathrm{E}[Q_1] + \sum_{i=2}^{k} \mathrm{E}[R_i],\\
G_{2(1,k)} &= \mathrm{E}[Q_{\mathrm{M}(1,k)}^2] = \mathrm{E}[Q_1^2] + \sum_{i=2}^{k} \mathrm{E}[R_i^2] + 2\sum_{i=2}^{k} \mathrm{E}[Q_1]\mathrm{E}[R_i] + 2\sum_{\substack{i=2\\ i<j}}^{k} \sum_{j=2}^{k} \mathrm{E}[R_i]\mathrm{E}[R_j],\\
\mathrm{E}[Q_1] &= \frac{N_k p_1}{N_k - p_1 - 2}, \quad \mathrm{E}[R_i] = \frac{N_{k-i+1} p_i}{N_{k-i+1} - p_{(12\ldots i)} - 2}, \quad i = 2, 3, \ldots, k,\\
\mathrm{E}[Q_1^2] &= \frac{N_k^2 p_1 (p_1 + 2)}{(N_k - p_1 - 2)(N_k - p_1 - 4)},\\
\mathrm{E}[R_i^2] &= \frac{N_{k-i+1}^2 p_i (p_i + 2)}{(N_{k-i+1} - p_{(12\ldots i)} - 2)(N_{k-i+1} - p_{(12\ldots i)} - 4)}, \quad i = 2, 3, \ldots, k.
\end{aligned}
$$

For the two-step monotone missing data case, see Onozawa et al. (2020). This approximation is based on the approximation of the upper percentile of the simplified T^2 statistic given by Krishnamoorthy and Pannala (1999).

2.3 Transformed test statistic

In this section, to improve the chi-squared approximation of the test statistic $Q_{\mathrm{M}(1,k)}$ proposed in Section 2.2, we consider a transformation with Bartlett adjustment based on Fujikoshi's approach (2000). Transformations for the simplified T^2 statistic $Q_{(1,k)}(= \sum_{i=1}^{k} Q_i)$ were given by Yagi et al.'s work (2019). In the same manner, transformations of $Q_{\mathrm{M}(1,k)}$ are considered. Because the transformed test statistics of Q_1 and R_i ($i = 2, 3, \ldots, k$) with Bartlett correction are given as

$$
Q_1^* = \left\{1 - \frac{1}{N_k}(p_1 + 2)\right\} Q_1, \quad R_i^* = \left\{1 - \frac{1}{N_{k-i+1}}(p_{(12\ldots i)} + 2)\right\} R_i, \quad i = 2, 3, \ldots, k,
$$

respectively, we propose the following statistic as an improvement of $Q_{\mathrm{M}(1,k)}$.

$$
Q_{\mathrm{M}(1,k)}^* = Q_1^* + \sum_{i=2}^{k} R_i^*. \tag{7}
$$

We note that $\mathrm{E}[Q_1^*] = p_1 + O(N_k^{-2})$, $\mathrm{E}[R_i^*] = p_i + O(N_{k-i+1}^{-2})$ and $\mathrm{E}[Q_{\mathrm{M}(1,k)}^*] = p + O(N_1^{-2})$. Further, we can obtain the transformed test statistics with Bartlett-type corrections as follows.

$$Y_1 = \left\{N_k - \frac{1}{2}(p_1+2)\right\}\log\left(1+\frac{1}{N_k}Q_1\right) \quad \text{for} \quad N_k - \frac{1}{2}(p_1+2) > 0,$$

$$Y_{iM} = \left\{N_{k-i+1} - \frac{1}{2}(2p_{(12\ldots,i-1)} + p_i + 2)\right\}\log\left(1+\frac{1}{N_{k-i+1}}R_i\right)$$

$$\text{for} \quad N_{k-i+1} - \frac{1}{2}(p_{(12\ldots,i-1)} + p_i + 2) > 0, \quad i = 2, 3, \ldots, k.$$

We note that $\Pr(Y_1 \leq x) = G_{p_1}(x) + O(N_k^{-2})$ and $\Pr(Y_{iM} \leq x) = G_{p_i}(x) + O(N_{k-i+1}^{-2})$. Using Y_1 and Y_{iM}, we propose a transformed test statistic given as

$$Y_{\mathrm{M}(1,k)} = Y_1 + \sum_{i=2}^{k} Y_{iM}. \tag{8}$$

Further, by using the result of the distribution function of $Q_{\mathrm{M}(1,k)}$ given in Theorem 1, we propose an exact Bartlett correction $Q^{\dagger}_{\mathrm{M}(1,k)}$ and an exact Bartlett-type correction $Y^{\dagger}_{\mathrm{M}(1,k)}$ of $Q_{\mathrm{M}(1,k)}$ as follows.

$$Q^{\dagger}_{\mathrm{M}(1,k)} = \left(1 - \frac{1}{n_1}c_1\right)Q_{\mathrm{M}(1,k)}, \tag{9}$$

$$Y^{\dagger}_{\mathrm{M}(1,k)} = (n_1 a + b)\log\left(1 + \frac{1}{n_1 a}Q_{\mathrm{M}(1,k)}\right) \quad (\text{for} \quad n_1 a + b > 0), \tag{10}$$

where

$$c_1 = \frac{1}{p}\sum_{i=1}^{k}\frac{1}{m_{k-i+1}}p_i(p_{(12\ldots i)}+2), \quad a = p(p+2)\left\{\sum_{i=1}^{k}\frac{1}{m_{k-i+1}}p_i(p_i+2)\right\}^{-1},$$

$$b = -\frac{p+2}{2}\left\{\frac{1}{m_k}p_1(p_1+2) + \sum_{i=2}^{k}\frac{1}{m_{k-i+1}}p_i(2p_{(12\ldots,i-1)}+p_i+2)\right\}$$
$$\times\left\{\sum_{i=1}^{k}\frac{1}{m_{k-i+1}}p_i(p_i+2)\right\}^{-1}.$$

Indeed, $\mathrm{E}[Q^{\dagger}_{\mathrm{M}(1,k)}] = p + O(n_1^{-2})$ and $\Pr(Y^{\dagger}_{\mathrm{M}(1,k)} \leq x) = G_p(x) + O(n_1^{-2})$.

3 Test statistics for two-sample problem

3.1 Test statistic similar to the simplified T^2 statistic

In this section, we consider the null distribution of a new test statistic similar to the simplified T^2 statistic for the two-sample problem when the datasets have k-step monotone missing data patterns.

Suppose two datasets

$$
\boldsymbol{X}^{(\ell)} = \begin{pmatrix}
\overbrace{\boldsymbol{X}_{11}^{(\ell)}}^{p_1} & \overbrace{\boldsymbol{X}_{12}^{(\ell)}}^{p_2} & \cdots & \overbrace{\boldsymbol{X}_{1,k-1}^{(\ell)}}^{p_{k-1}} & \overbrace{\boldsymbol{X}_{1k}^{(\ell)}}^{p_k} \\
\boldsymbol{X}_{21}^{(\ell)} & \boldsymbol{X}_{22}^{(\ell)} & \cdots & \boldsymbol{X}_{2,k-1}^{(\ell)} & * \\
\vdots & \vdots & & & \vdots \\
\boldsymbol{X}_{k-1,1}^{(\ell)} & \boldsymbol{X}_{k-1,2}^{(\ell)} & * & \cdots & * \\
\boldsymbol{X}_{k1}^{(\ell)} & * & \cdots & \cdots & *
\end{pmatrix}
\begin{matrix} \}n_1^{(\ell)} \\ \}n_2^{(\ell)} \\ \\ \}n_{k-1}^{(\ell)} \\ \}n_k^{(\ell)} \end{matrix}, \quad \ell = 1, 2,
$$

are independent and distributed as multivariate normal distribution with a common covariance matrix, where $\boldsymbol{X}_{ij}^{(\ell)}$ is a $n_i^{(\ell)} \times p_j$ block matrix $(i = 1, 2, \ldots, k; j = 1, 2, \ldots, k - i + 1)$.

As with the one-sample problem in Section 2.2, let

$$
\boldsymbol{X}_{i(12\ldots,k-i+1)}^{(\ell)} = (\boldsymbol{X}_{i1}^{(\ell)} \;\; \boldsymbol{X}_{i2}^{(\ell)} \;\; \cdots \;\; \boldsymbol{X}_{i,k-i+1}^{(\ell)}).
$$

Then, we assume that the rows of $\boldsymbol{X}_{i(12\ldots,k-i+1)}^{(\ell)}$ $(i = 1, 2, \ldots, k)$ are mutually independent and that

$$
\mathrm{vec}(\boldsymbol{X}_{i(12\ldots,k-i+1)}^{(\ell)\prime}) \sim N_{n_i^{(\ell)} p_{(12\ldots,k-i+1)}}(\mathrm{vec}(\boldsymbol{\mu}_{(12\ldots,k-i+1)}^{(\ell)} \mathbf{1}'_{n_i^{(\ell)}}), \boldsymbol{I}_{n_i^{(\ell)}} \otimes \boldsymbol{\Sigma}_{(12\ldots,k-i+1)(12\ldots,k-i+1)}),
$$
$$
i = 1, 2, \ldots, k,
$$

where

$$
\underset{p_{(12\ldots,k-i+1)} \times 1}{\boldsymbol{\mu}_{(12\ldots,k-i+1)}^{(\ell)}} = \begin{pmatrix} \boldsymbol{\mu}_1^{(\ell)} \\ \boldsymbol{\mu}_2^{(\ell)} \\ \vdots \\ \boldsymbol{\mu}_{k-i+1}^{(\ell)} \end{pmatrix} \begin{matrix} \}p_1 \\ \}p_2 \\ \\ \}p_{k-i+1} \end{matrix}.
$$

Further, we define

$$
\begin{aligned}
\boldsymbol{X}_{(12\ldots,k-i+1)(12\ldots i)}^{(\ell)} &= \begin{pmatrix} \boldsymbol{X}_{11}^{(\ell)} & \cdots & \boldsymbol{X}_{1i}^{(\ell)} \\ \vdots & & \vdots \\ \boldsymbol{X}_{k-i+1,1}^{(\ell)} & \cdots & \boldsymbol{X}_{k-i+1,i}^{(\ell)} \end{pmatrix}, \\
\overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots i)}^{(\ell)} &= \frac{1}{N_{k-i+1}^{(\ell)}} \boldsymbol{X}_{(12\ldots,k-i+1)(12\ldots i)}^{(\ell)\prime} \mathbf{1}_{N_{k-i+1}^{(\ell)}}, \\
\boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i)}^{(\ell)} &= \frac{1}{N_{k-i+1}^{(\ell)} - 1} \Big\{ \boldsymbol{X}_{(12\ldots,k-i+1)(12\ldots i)}^{(\ell)\prime} \boldsymbol{X}_{(12\ldots,k-i+1)(12\ldots i)}^{(\ell)} \\
&\quad - N_{k-i+1}^{(\ell)} \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots i)}^{(\ell)} \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots i)}^{(\ell)\prime} \Big\}, \quad i = 1, 2, \ldots, k,
\end{aligned}
$$

$$\overline{\boldsymbol{x}}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots i)} = \begin{pmatrix} \overline{\boldsymbol{x}}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots,i-1)} \\ \overline{\boldsymbol{x}}^{(\ell)}_{(12\ldots,k-i+1)i} \end{pmatrix} \begin{matrix} \}p_{(12\ldots,i-1)} \\ \}p_i \end{matrix},$$

$$\boldsymbol{S}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots i)} = \begin{pmatrix} \overbrace{\boldsymbol{S}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots i),11}}^{p_{(12\ldots,i-1)}} & \overbrace{\boldsymbol{S}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots i),12}}^{p_i} \\ \boldsymbol{S}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots i),21} & \boldsymbol{S}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots i),22} \end{pmatrix} \begin{matrix} \}p_{(12\ldots,i-1)} \\ \}p_i \end{matrix}, \quad i = 2, 3, \ldots, k,$$

where $N^{(\ell)}_{k-i+1} = \sum_{j=1}^{k-i+1} n^{(\ell)}_j$.

Then, a test statistic for the hypothesis

$$H_0 : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} \quad \text{vs.} \quad H_1 : \boldsymbol{\mu}^{(1)} \neq \boldsymbol{\mu}^{(2)},$$

is given by

$$Q_{\mathrm{M}(2,k)} = Q_1 + \sum_{i=2}^{k} R_i, \tag{11}$$

where

$$\begin{aligned}
Q_i =& \frac{N^{(1)}_{k-i+1} N^{(2)}_{k-i+1}}{N^{(1)}_{k-i+1} + N^{(2)}_{k-i+1}} (\widehat{\boldsymbol{\eta}}^{(1)}_i - \widehat{\boldsymbol{\eta}}^{(2)}_i)' \widehat{\boldsymbol{\Delta}}^{-1}_{iip\ell} (\widehat{\boldsymbol{\eta}}^{(1)}_i - \widehat{\boldsymbol{\eta}}^{(2)}_i), \quad i = 1, 2, \ldots, k, \\
R_i =& \frac{Q_i}{1 + Q_{id}}, \\
Q_{id} =& \frac{N^{(1)}_{k-i+1} N^{(2)}_{k-i+1}}{N^{(1)}_{k-i+1} + N^{(2)}_{k-i+1}} (\overline{\boldsymbol{x}}^{(1)}_{(12\ldots,k-i+1)(12\ldots,i-1)} - \overline{\boldsymbol{x}}^{(2)}_{(12\ldots,k-i+1)(12\ldots,i-1)})' \boldsymbol{W}^{-1}_{k-i+1,11} \\
& \times (\overline{\boldsymbol{x}}^{(1)}_{(12\ldots,k-i+1)(12\ldots,i-1)} - \overline{\boldsymbol{x}}^{(2)}_{(12\ldots,k-i+1)(12\ldots,i-1)}), \quad i = 2, 3, \ldots, k, \\
\widehat{\boldsymbol{\eta}}^{(\ell)}_1 =& \overline{\boldsymbol{x}}^{(\ell)}_{(12\ldots k)1}, \quad \widehat{\boldsymbol{\Delta}}_{11p\ell} = \frac{1}{N^{(1)}_k + N^{(2)}_k} \boldsymbol{W}_k, \\
\widehat{\boldsymbol{\eta}}^{(\ell)}_i =& \overline{\boldsymbol{x}}^{(\ell)}_{(12\ldots,k-i+1)i} - \boldsymbol{W}_{k-i+1,21} \boldsymbol{W}^{-1}_{k-i+1,11} \overline{\boldsymbol{x}}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots,i-1)}, \\
\widehat{\boldsymbol{\Delta}}_{iip\ell} =& \frac{1}{N^{(1)}_{k-i+1} + N^{(2)}_{k-i+1}} \boldsymbol{W}_{k-i+1,22.1}, \quad i = 2, 3, \ldots, k, \\
\boldsymbol{W}_k =& \sum_{\ell=1}^{2} (N^{(\ell)}_k - 1) \boldsymbol{S}^{(\ell)}_{(12\ldots k)1}, \\
\boldsymbol{W}_{k-i+1,jq} =& \sum_{\ell=1}^{2} (N^{(\ell)}_{k-i+1} - 1) \boldsymbol{S}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots i),jq}, \quad j = 1, 2; q = 1, 2, \\
\boldsymbol{W}_{k-i+1,22.1} =& \boldsymbol{W}_{k-i+1,22} - \boldsymbol{W}_{k-i+1,21} \boldsymbol{W}^{-1}_{k-i+1,11} \boldsymbol{W}_{k-i+1,12}, \quad i = 2, 3, \ldots, k.
\end{aligned}$$

Then, as with the one-sample problem, we obtain the following theorem.

Theorem 3

For large ν_1*, the characteristic function of* $Q_{\mathrm{M}(2,k)}$ *can be expanded as*

$$\mathrm{E}[\exp(itQ_{\mathrm{M}(2,k)})] = (1-2it)^{-\frac{1}{2}p} + \frac{1}{\nu_1}\sum_{j=0}^{2}\gamma_j(1-2it)^{-\frac{1}{2}p-j} + O(\nu_1^{-2}),$$

where

$$\gamma_j = \frac{1}{(\sum_{j=1}^{k} s_j)}\beta_{j,1} + \sum_{i=2}^{k}\frac{1}{(\sum_{j=1}^{k-i+1} s_j)}\gamma_{j,i}, \quad j=0,1,2, \quad \beta_{0,1} = -\frac{1}{4}p_1(p_1+4),$$
$$\beta_{1,1} = \frac{1}{2}p_1, \quad \beta_{2,1} = \frac{1}{4}p_1(p_1+2), \quad \gamma_{0,i} = -\frac{1}{4}p_i(2p_{(12\dots,i-1)} + p_i + 4),$$
$$\gamma_{1,i} = \frac{1}{2}p_i(p_{(12\dots,i-1)}+1), \quad \gamma_{2,i} = \frac{1}{4}p_i(p_i+2), \quad s_i = \frac{\nu_i}{\nu_1}, \quad \nu_i = \sum_{\ell=1}^{2} n_i^{(\ell)}, \quad i=1,2,\dots,k,$$

and s_i *is a positive constant. Also, the distribution function of* $Q_{\mathrm{M}(2,k)}$ *is*

$$\Pr(Q_{\mathrm{M}(2,k)} \le x) = G_p(x) + \frac{1}{\nu_1}\sum_{j=0}^{2}\gamma_j G_{p+2j}(x) + O(\nu_1^{-2}),$$

where $G_f(x)$ *is the distribution function of a chi-squared variate with* f *degrees of freedom.*

The derivation of Theorem 3 is provided in the Appendix. From the result of Theorem 3, its upper 100α percentiles can be expanded as

$$q_{\mathrm{M}(2,k)}(\alpha) = \chi_p^2(\alpha) - \frac{1}{\nu_1}\left[\frac{2\chi_p^2(\alpha)}{p}\left\{\gamma_0 - \frac{\gamma_2}{p+2}\chi_p^2(\alpha)\right\}\right] + O(\nu_1^{-2}),$$

where $\chi_p^2(\alpha)$ is the upper 100α percentile of chi-squared distribution with p degrees of freedom. Therefore, an approximation to the upper 100α percentile of $Q_{\mathrm{M}(2,k)}$ is given by

$$q_{\mathrm{M}(2,k)\mathrm{AE}}(\alpha) = \chi_p^2(\alpha) - \frac{1}{\nu_1}\frac{2\chi_p^2(\alpha)}{p}\left\{\gamma_0 - \frac{\gamma_2}{p+2}\chi_p^2(\alpha)\right\}. \tag{12}$$

As another approximate upper 100α percentile of $Q_{\mathrm{M}(2,k)}$, in the same manner as for the one-sample problem, we propose

$$q_{\mathrm{M}(2,k)\mathrm{YKP}}(\alpha) = d_{(2,k)}F_{p,\nu_{(2,k)}}(\alpha) \quad (\text{for } \nu_1 > p+5), \tag{13}$$

where $F_{p,\nu_{(2,k)}}(\alpha)$ is the upper 100α percentile of F-distribution with p and $\nu_{(2,k)}$ degrees of freedom,

$$
\begin{aligned}
d_{(2,k)} &= G_{1(2,k)} \frac{\nu_{(2,k)} - 2}{\nu_{(2,k)}}, \quad \nu_{(2,k)} = \frac{4pG_{2(2,k)} - 2(p+2)G_{1(2,k)}^2}{pG_{2(2,k)} - (p+2)G_{1(2,k)}^2}, \\
G_{1(2,k)} &= \mathrm{E}[Q_{\mathrm{M}(2,k)}] = \mathrm{E}[Q_1] + \sum_{i=2}^{k} \mathrm{E}[R_i], \\
G_{2(2,k)} &= \mathrm{E}[Q_{\mathrm{M}(2,k)}^2] = \mathrm{E}[Q_1^2] + \sum_{i=2}^{k} \mathrm{E}[R_i^2] + 2\sum_{i=2}^{k} \mathrm{E}[Q_1]\mathrm{E}[R_i] + 2\sum_{\substack{i=2 \\ i<j}}^{k}\sum_{j=2}^{k} \mathrm{E}[R_i]\mathrm{E}[R_j], \\
\mathrm{E}[Q_1] &= \frac{M_k p_1}{M_k - p_1 - 3}, \quad \mathrm{E}[R_i] = \frac{M_{k-i+1} p_i}{M_{k-i+1} - p_{(12\ldots i)} - 3}, \quad i = 2, 3, \ldots, k, \\
\mathrm{E}[Q_1^2] &= \frac{M_k^2 p_1 (p_1 + 2)}{(M_k - p_1 - 3)(M_k - p_1 - 5)}, \\
\mathrm{E}[R_i^2] &= \frac{M_{k-i+1}^2 p_i (p_i + 2)}{(M_{k-i+1} - p_{(12\ldots i)} - 3)(M_{k-i+1} - p_{(12\ldots i)} - 5)}, \quad i = 2, 3, \ldots, k.
\end{aligned}
$$

Onozawa et al. (2020) also proposed the approximation $q_{\mathrm{M}(2,2)\mathrm{YKP}}(\alpha)$ for datasets with two-step monotone missing data patterns. This approximation is based on the approximation by Yu, Krishnamoorthy and Pannala (2006).

Yu et al. (2006) and Yagi et al. (2023) discussed the simplified T^2 statistic for the two-sample case as given below.

$$
Q_{(2,k)} = \sum_{i=1}^{k} Q_i. \tag{14}
$$

We describe the results of a Monte Carlo simulation executed to compare the tests $Q_{\mathrm{M}(2,k)}$ and $Q_{(2,k)}$ in Section 5.

3.2 Transformed test statistic

In this section, as with the one-sample problem, we consider transformations of $Q_{\mathrm{M}(2,k)}$. The transformed test statistics of Q_1 and R_i $(i = 2, 3, \ldots, k)$ with Bartlett correction for the two-sample problem are given by

$$
Q_1^* = \left\{1 - \frac{1}{M_k}(p_1 + 3)\right\} Q_1 \text{ and } R_i^* = \left\{1 - \frac{1}{M_{k-i+1}}(p_{(12\ldots i)} + 3)\right\} R_i, \quad i = 2, 3, \ldots, k,
$$

respectively, where $M_i = \sum_{\ell=1}^{2} N_i^{(\ell)}$. Thus, as an improvement of $Q_{\mathrm{M}(2,k)}$, we propose

$$
Q_{\mathrm{M}(2,k)}^* = Q_1^* + \sum_{i=2}^{k} R_i^*. \tag{15}
$$

We note that $\mathrm{E}[Q_1^*] = p_1 + O(M_k^{-2})$, $\mathrm{E}[R_i^*] = p_i + O(M_{k-i+1}^{-2})$ and $\mathrm{E}[Q_{\mathrm{M}(2,k)}^*] = p + O(M_1^{-2})$. We can also obtain the transformed test statistics of Q_1 and R_i $(i = 2, 3, \ldots, k)$ with Bartlett-type corrections as follows.

$$
\begin{aligned}
Y_1 &= \left\{M_k - \frac{1}{2}(p_1+4)\right\} \log\left(1 + \frac{1}{M_k} Q_1\right) \quad \text{for} \quad M_k - \frac{1}{2}(p_1+2) > 0, \\
Y_{iM} &= \left\{M_{k-i+1} - \frac{1}{2}(2p_{(12\ldots,i-1)} + p_i + 4)\right\} \log\left(1 + \frac{1}{M_{k-i+1}} R_i\right) \\
&\qquad \text{for} \quad M_{k-i+1} - \frac{1}{2}(p_{(12\ldots,i-1)} + p_i + 4) > 0, \quad i = 2, 3, \ldots, k.
\end{aligned}
$$

We note that $\Pr(Y_1 \le x) = G_{p_1}(x) + O(M_k^{-2})$ and $\Pr(Y_{iM} \le x) = G_{p_i}(x) + O(M_{k-i+1}^{-2})$. Then, we propose a transformed test statistic as

$$
Y_{\mathrm{M}(2,k)} = Y_1 + \sum_{i=2}^{k} Y_{iM}. \tag{16}
$$

Further, from the result of the distribution function of $Q_{\mathrm{M}(1,k)}$ given in Theorem 3, we can propose an exact Bartlett correction $Q_{\mathrm{M}(2,k)}^{\dagger}$ and an exact Bartlett-type correction $Y_{\mathrm{M}(2,k)}^{\dagger}$ of $Q_{\mathrm{M}(2,k)}$, which satisfies $\mathrm{E}[Q_{\mathrm{M}(2,k)}^{\dagger}] = p + O(M_1^{-2})$ and $\Pr(Y_{\mathrm{M}(2,k)}^{\dagger} \le x) = G_p(x) + O(M_1^{-2})$, where

$$
Q_{\mathrm{M}(2,k)}^{\dagger} = \left(1 - \frac{1}{M_1} c_1\right) Q_{\mathrm{M}(2,k)}, \tag{17}
$$

$$
Y_{\mathrm{M}(2,k)}^{\dagger} = (M_1 a + b) \log\left(1 + \frac{1}{M_1 a} Q_{\mathrm{M}(2,k)}\right) \quad (\text{for} \quad M_1 a + b > 0), \tag{18}
$$

with

$$
\begin{aligned}
c_1 &= \frac{1}{p} \sum_{i=1}^{k} \frac{1}{\sum_{j=1}^{k-i+1} s_j} p_i (p_{(12\ldots i)} + 3), \quad a = p(p+2) \left\{\sum_{i=1}^{k} \frac{1}{\sum_{j=1}^{k-i+1} s_j} p_i (p_i + 2)\right\}^{-1}, \\
b &= -\frac{p+2}{2} \left\{\frac{1}{\sum_{j=1}^{k} s_j} p_1(p_1+4) + \sum_{i=2}^{k} \frac{1}{\sum_{j=1}^{k-i+1} s_j} p_i (2p_{(12\ldots,i-1)} + p_i + 4)\right\} \\
&\quad \times \left\{\sum_{i=1}^{k} \frac{1}{\sum_{j=1}^{k-i+1} s_j} p_i (p_i + 2)\right\}^{-1}.
\end{aligned}
$$

4 Multivariate pairwise comparisons among mean vectors

In this section, we consider the simultaneous confidence intervals for multivariate pairwise comparisons among mean vectors when each dataset has k-step monotone missing observations. Let

$$\boldsymbol{X}^{(\ell)} = \begin{pmatrix} \overbrace{\boldsymbol{X}_{11}^{(\ell)}}^{p_1} & \overbrace{\boldsymbol{X}_{12}^{(\ell)}}^{p_2} & \cdots & \overbrace{\boldsymbol{X}_{1,k-1}^{(\ell)}}^{p_{k-1}} & \overbrace{\boldsymbol{X}_{1k}^{(\ell)}}^{p_k} \\ \boldsymbol{X}_{21}^{(\ell)} & \boldsymbol{X}_{22}^{(\ell)} & \cdots & \boldsymbol{X}_{2,k-1}^{(\ell)} & * \\ \vdots & \vdots & & & \vdots \\ \boldsymbol{X}_{k-1,1}^{(\ell)} & \boldsymbol{X}_{k-1,2}^{(\ell)} & * & \cdots & * \\ \boldsymbol{X}_{k1}^{(\ell)} & * & \cdots & \cdots & * \end{pmatrix} \begin{matrix} \}n_1^{(\ell)} \\ \}n_2^{(\ell)} \\ \\ \}n_{k-1}^{(\ell)} \\ \}n_k^{(\ell)} \end{matrix}, \quad \ell = 1, 2, \ldots, m.$$

For other settings, replace $\ell = 1, 2$ with $\ell = 1, 2, \ldots, m$ in Section 3.1.

In Yagi and Seo (2017), for the case of pairwise multiple comparisons, the simultaneous confidence intervals for $\boldsymbol{c}'(\boldsymbol{\mu}^{(a)} - \boldsymbol{\mu}^{(b)}), \forall \boldsymbol{c} \in \boldsymbol{R}^p - \{\boldsymbol{0}\}$, $1 \leq a < b \leq m$ with the confidence level $(1 - \alpha)$ are given by

$$\boldsymbol{c}'(\boldsymbol{\mu}^{(a)} - \boldsymbol{\mu}^{(b)}) \in \left[\boldsymbol{c}'(\widehat{\boldsymbol{\mu}}^{(a)} - \widehat{\boldsymbol{\mu}}^{(b)}) \pm (t_{\max}^2(\alpha)\boldsymbol{c}'\widetilde{\boldsymbol{\Gamma}}\boldsymbol{c})^{\frac{1}{2}}\right], \ \forall \boldsymbol{c} \in \boldsymbol{R}^p - \{\boldsymbol{0}\}, \ 1 \leq a < b \leq m,$$

where $t_{\max}^2(\alpha)$ is the upper 100α percentile of $T_{\max}^2$,

$$T_{\max}^2 = \max_{1 \leq a < b \leq m} Q^{(ab)}, \quad Q^{(ab)} = (\widehat{\boldsymbol{\mu}}^{(a)} - \widehat{\boldsymbol{\mu}}^{(b)})'\widetilde{\boldsymbol{\Gamma}}^{-1}(\widehat{\boldsymbol{\mu}}^{(a)} - \widehat{\boldsymbol{\mu}}^{(b)}),$$

$\widehat{\boldsymbol{\mu}}^{(\ell)}$ $(\ell = a, b)$ is MLE of $\boldsymbol{\mu}^{(\ell)}$, $\widetilde{\boldsymbol{\Gamma}}$ is an estimator of $\mathrm{Cov}(\widetilde{\boldsymbol{\mu}}^{(a)} - \widetilde{\boldsymbol{\mu}}^{(b)})$, and $\widetilde{\boldsymbol{\mu}}^{(\ell)}$ is an MLE of $\boldsymbol{\mu}^{(\ell)}$ when $\boldsymbol{\Sigma}$ is known. Then, the upper percentile $t_{\max}^2(\alpha)$ is needed to construct the approximate simultaneous confidence intervals for pairwise comparisons among mean vectors. Owing to the difficulty in finding it precisely, Yagi and Seo (2017) gave an approximation of $t_{\max}^2(\alpha)$ by linear interpolation. Also, Yagi, Seo, and Hanusz (2018) gave an approximation of $t_{\max}^2(\alpha)$ by decomposing $Q^{(ab)} = \sum_{i=1}^k Q_i^{(ab)}$ and using the asymptotic expansion of each term.

In the present work, we propose another new approximation. For $1 \leq a < b \leq m$, as with the one-sample and two-sample cases considered in previous sections, we define $Q_{\mathrm{M}}^{(ab)}$ and consider stochastic expansion of each term, where

$$Q_{\mathrm{M}}^{(ab)} = Q_1^{(ab)} + \sum_{i=2}^k R_i^{(ab)},$$

where

$$
\begin{aligned}
Q_i^{(ab)} =& \frac{N_{k-i+1}^{(a)} N_{k-i+1}^{(b)}}{N_{k-i+1}^{(a)} + N_{k-i+1}^{(b)}} (\widehat{\boldsymbol{\zeta}}_i^{(a)} - \widehat{\boldsymbol{\zeta}}_i^{(b)})' \widehat{\boldsymbol{\Psi}}_{ii}^{-1} (\widehat{\boldsymbol{\zeta}}_i^{(a)} - \widehat{\boldsymbol{\zeta}}_i^{(b)}), \quad i = 1, 2, \ldots, k, \\
R_i^{(ab)} =& \frac{Q_i^{(ab)}}{1 + Q_{id}^{(ab)}}, \\
Q_{id}^{(ab)} =& \frac{N_{k-i+1}^{(a)} N_{k-i+1}^{(b)}}{N_{k-i+1}^{(a)} + N_{k-i+1}^{(b)}} (\overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots,i-1)}^{(a)} - \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots,i-1)}^{(b)})' \boldsymbol{U}_{k-i+1,11}^{-1} \\
& \times (\overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots,i-1)}^{(a)} - \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots,i-1)}^{(b)}), \quad i = 2, 3, \ldots, k, \\
\widehat{\boldsymbol{\zeta}}_1^{(\ell)} =& \overline{\boldsymbol{x}}_{(12\ldots k)1}^{(\ell)}, \ \widehat{\boldsymbol{\zeta}}_i^{(\ell)} = \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)i}^{(\ell)} - \boldsymbol{U}_{k-i+1,21} \boldsymbol{U}_{k-i+1,11}^{-1} \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots,i-1)}^{(\ell)}, \ \ell = a, b, \\
\widehat{\boldsymbol{\Psi}}_{11} =& \frac{1}{M_k} \boldsymbol{U}_k, \quad \widehat{\boldsymbol{\Psi}}_{ii} = \frac{1}{M_{k-i+1}} \boldsymbol{U}_{k-i+1,22.1}, \quad i = 2, 3, \ldots, k, \\
M_i =& \sum_{\ell=1}^m N_i^{(\ell)}, \quad i = 1, 2, \ldots, k, \quad \boldsymbol{U}_k = \sum_{\ell=1}^m (N_k^{(\ell)} - 1) \boldsymbol{S}_{(12\ldots k)1}^{(\ell)}, \\
\boldsymbol{U}_{k-i+1,jq} =& \sum_{\ell=1}^m (N_{k-i+1}^{(\ell)} - 1) \boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i),jq}^{(\ell)}, \quad j = 1, 2; q = 1, 2, \\
\boldsymbol{U}_{k-i+1,22.1} =& \boldsymbol{U}_{k-i+1,22} - \boldsymbol{U}_{k-i+1,21} \boldsymbol{U}_{k-i+1,11}^{-1} \boldsymbol{U}_{k-i+1,12}, \quad i = 2, 3, \ldots, k.
\end{aligned}
$$

If we adopt the same approach as in the two-sample problem, the distribution of $Q_{\mathrm{M}}^{(ab)}$ and the upper 100α percentile of $Q_{\mathrm{M}}^{(ab)}$, $q_{\mathrm{MAE}}(\alpha)$ are given by the following theorem, and we use $q_{\mathrm{MAE}}(\alpha_{\mathrm{p}})$ as an approximation to $t_{\max}^2(\alpha)$, where $\alpha_{\mathrm{p}} = 2\alpha/\{m(m-1)\}$ and $n_i^{(1)} = n_i^{(2)} = \cdots = n_i^{(m)}$, $i = 1, 2, \ldots, k$ (using Bonferroni's inequality).

Theorem 4

For large ν_1, the characteristic function of $Q_{\mathrm{M}}^{(ab)}$ can be expanded as

$$
\mathrm{E}[\exp(itQ_{\mathrm{M}}^{(ab)})] = (1-2it)^{-\frac{1}{2}p} + \frac{1}{\nu_1} \sum_{j=0}^{2} \gamma_j (1-2it)^{-\frac{1}{2}p-j} + O(\nu_1^{-2}),
$$

where

$$
\begin{aligned}
\gamma_0 &= -\frac{1}{4} \sum_{i=1}^k \left[\frac{1}{(\sum_{j=1}^{k-i+1} s_j)} p_i (2p_{(12\ldots,i-1)} + p_i + 2m) \right], \\
\gamma_1 &= \frac{1}{2} \sum_{i=1}^k \left[\frac{1}{(\sum_{j=1}^{k-i+1} s_j)} p_i (p_{(12\ldots,i-1)} + m - 1) \right], \quad \gamma_2 = \frac{1}{4} \sum_{i=1}^k \left[\frac{1}{(\sum_{j=1}^{k-i+1} s_j)} p_i (p_i + 2) \right], \\
s_i &= \frac{\nu_i}{\nu_1}, \quad \nu_i = \sum_{\ell=1}^m n_i^{(\ell)}, \quad i = 1, 2, \ldots, k,
\end{aligned}
$$

and s_i is a positive constant. Also, the distribution function of $Q_{\mathrm{M}}^{(ab)}$ is given as

$$\Pr(Q_{\mathrm{M}}^{(ab)} \leq x) = G_p(x) + \frac{1}{\nu_1}\sum_{j=0}^{2}\gamma_j G_{p+2j}(x) + O(\nu_1^{-2}),$$

where $G_f(x)$ is the distribution function of a chi-squared variate with f degrees of freedom.

From the result of Theorem 4, its upper 100α percentiles can be expanded as

$$q_{\mathrm{M}}^{(ab)}(\alpha) = \chi_p^2(\alpha) - \frac{1}{\nu_1}\left[\frac{2\chi_p^2(\alpha)}{p}\left\{\gamma_0 - \frac{\gamma_2}{p+2}\chi_p^2(\alpha)\right\}\right] + O(\nu_1^{-2}),$$

where $\chi_p^2(\alpha)$ is the upper 100α percentile of chi-squared distribution with p degrees of freedom. Therefore, an approximation to the upper $100\alpha_{\mathrm{p}}$ percentile of $Q_{\mathrm{M}}^{(ab)}$ is given by

$$q_{\mathrm{MAE}}(\alpha_{\mathrm{p}}) = \chi_p^2(\alpha_{\mathrm{p}}) - \frac{1}{\nu_1}\frac{2\chi_p^2(\alpha)_{\mathrm{p}}}{p}\left\{\gamma_0 - \frac{\gamma_2}{p+2}\chi_p^2(\alpha_{\mathrm{p}})\right\}, \tag{19}$$

which we use as an approximation to $t_{\max}^2(\alpha)$. Approximate simultaneous confidence intervals for comparisons with a control can be obtained by replacing α_{p} in the previous discussion with α_{c}, where $\alpha_{\mathrm{c}} = \alpha/(m-1)$.

5 Simulation studies

In this section, we describe the procedure and results of simulations conducted to examine the accuracy of the proposed approximations and the asymptotic behavior of the statistics and approximate upper percentiles for one- and two-sample problems. For the pairwise comparisons, we evaluated the accuracy of the approximate upper percentile ($q_{\mathrm{MAE}}(\alpha_{\mathrm{p}})$) proposed in Section 4 as an approximation of the upper percentile of $T_{\max}^2$.

For one-sample and two-sample cases, we computed the upper percentiles of the test statistics $Q_{\mathrm{M}(1,k)}$ in (4) and $Q_{\mathrm{M}(2,k)}$ in (11) and the transformed test statistics $Q_{\mathrm{M}(\ell,k)}^{*}$, $Y_{\mathrm{M}(\ell,k)}$, $Q_{\mathrm{M}(\ell,k)}^{\dagger}$ and $Y_{\mathrm{M}(\ell,k)}^{\dagger}$ in Section 2.3 ($\ell = 1$) and Section 3.2 ($\ell = 2$) with k-step monotone missing data by Monte Carlo simulation. The simulation was executed 10^6 times using normal random vectors generated from $N_{p_{(12\ldots i)}}(\mathbf{0}, \boldsymbol{I}_{p_{(12\ldots i)}})$, $i = 1, 2, \ldots, k$ for selected parameters. The results of the simulations for the upper percentiles of the test statistics, their approximations, and their empirical type I errors are provided in the Tables below. The upper 100α percentiles and empirical type I error rates are denoted as follows.

(i) q_{M} is the upper 100α percentile of $Q_{\mathrm{M}(\ell,k)}$ ($\ell = 1$ or 2) by simulation, and

$$\frac{\alpha_{\mathrm{M}\chi^2}}{100} = \Pr(Q_{\mathrm{M}(\ell,k)} > \chi_p^2(\alpha)) \quad (\ell = 1 \text{ or } 2).$$

(ii) q_{MAE} is $q_{\mathrm{M}(\ell,k)\mathrm{AE}}(\alpha)$ given in (5) ($\ell = 1$) or (12) ($\ell = 2$), and

$$\frac{\alpha_{\mathrm{MAE}}}{100} = \Pr(Q_{\mathrm{M}(\ell,k)} > q_{\mathrm{M}(\ell,k)\mathrm{AE}}(\alpha)) \quad (\ell = 1 \text{ or } 2).$$

(iii) q_{MKP} and q_{MYKP} are $q_{\mathrm{M}(1,k)\mathrm{KP}}(\alpha)$ given in (6) and $q_{\mathrm{M}(2,k)\mathrm{YKP}}(\alpha)$ given in (13), respectively. Further,

$$\frac{\alpha_{\mathrm{MKP}}}{100} = \Pr(Q_{\mathrm{M}(1,k)} > q_{\mathrm{M}(1,k)\mathrm{KP}}(\alpha)) \text{ and } \frac{\alpha_{\mathrm{MYKP}}}{100} = \Pr(Q_{\mathrm{M}(2,k)} > q_{\mathrm{M}(2,k)\mathrm{YKP}}(\alpha)).$$

(iv) q^*_{M} is the upper 100α percentile of $Q^*_{\mathrm{M}(\ell,k)}$ in (7) ($\ell = 1$) or (15) ($\ell = 2$) by simulation, and

$$\frac{\alpha^*_{\mathrm{M}}}{100} = \Pr(Q^*_{\mathrm{M}(\ell,k)} > \chi_p^2(\alpha)) \quad (\ell = 1 \text{ or } 2).$$

(v) $q^\dagger_{\mathrm{M}}$ is the upper 100α percentile of $Q^\dagger_{\mathrm{M}(\ell,k)}$ in (9) ($\ell = 1$) or (17) ($\ell = 2$) by simulation, and

$$\frac{\alpha^\dagger_{\mathrm{M}}}{100} = \Pr(Q^\dagger_{\mathrm{M}(\ell,k)} > \chi_p^2(\alpha)) \quad (\ell = 1 \text{ or } 2).$$

(vi) $q_{Y_{\mathrm{M}}}$ is the upper 100α percentile of $Y_{\mathrm{M}(\ell,k)}$ in (8) ($\ell = 1$) or (16) ($\ell = 2$) by simulation, and

$$\frac{\alpha_{Y_{\mathrm{M}}}}{100} = \Pr(Y_{\mathrm{M}(\ell,k)} > \chi_p^2(\alpha)) \quad (\ell = 1 \text{ or } 2).$$

(vii) $q^\dagger_{Y_{\mathrm{M}}}$ is the upper 100α percentile of $Y^\dagger_{\mathrm{M}(\ell,k)}$ in (10) ($\ell = 1$) or (18) ($\ell = 2$) by simulation, and

$$\frac{\alpha^\dagger_{Y_{\mathrm{M}}}}{100} = \Pr(Y^\dagger_{\mathrm{M}(\ell,k)} > \chi_p^2(\alpha)) \quad (\ell = 1 \text{ or } 2).$$

Further, we compare these upper percentiles and empirical type I error rates with its counterparts of the simplified T^2 statistics in Yagi et al. (2019, 2023). To save space, only q, q_{KP} (or q_{YKP}), q^*, q_Y, α_{χ^2}, α_{KP} (or α_{YKP}), α^*, and α_Y are listed, where q, q^*, q_Y are the upper 100α percentile of $Q_{(\ell,k)}$, $Q^*_{(\ell,k)}$, $Y_{(\ell,k)}$ ($\ell = 1$, or 2) by simulation, respectively. $Q^*_{(\ell,k)}$ and $Y_{(\ell,k)}$ ($\ell = 1$, or 2) are the test statistics Q^* and Y proposed by Yagi et al. (2019, 2023). q_{KP}, q_{YKP} are an approximation to the upper 100α percentile of $Q_{(1,k)}$ in

(3) given by Krishnamoorthy and Pannala (1999) and $Q_{(2,k)}$ in (14) given by Yu et al. (2006), respectively. Further

$$\frac{\alpha_{\chi^2}}{100} = \Pr(Q_{(\ell,k)} > \chi_p^2(\alpha)),\ \frac{\alpha^*}{100} = \Pr(Q^*_{(\ell,k)} > \chi_p^2(\alpha)),\ \frac{\alpha_Y}{100} = \Pr(Y_{(\ell,k)} > \chi_p^2(\alpha)),\ \ell = 1 \text{ or } 2,$$
$$\frac{\alpha_{\mathrm{KP}}}{100} = \Pr(Q_{(1,k)} > q_{\mathrm{KP}}), \quad \text{and} \quad \frac{\alpha_{\mathrm{YKP}}}{100} = \Pr(Q_{(2,k)} > q_{\mathrm{YKP}}).$$

Tables 1-4 present the results in the case of $k = 3, 5; \ell = 1, 2$ with $p_i = 2, i = 1, 2, \ldots, k$. It may be noted from the tables that the value of $q_{Y_{\mathrm{M}}}$ converged to the upper percentile of the chi-squared limiting distribution very rapidly. In addition, the values of q_{MKP} and q_{MYKP} were very good approximations for most cases. In particular, for $\ell = 1, 2$, $Y_{(\ell,k)}$ became conservative when the sample size was very small, whereas $Y_{\mathrm{M}(\ell,k)}$ exhibited very good approximation accuracy even when the sample size was very small. The values of $\alpha_{\mathrm{M}\chi^2}$, α_{MAE}, α_{MKP} (or α_{MYKP}), α^*_{M}, $\alpha^\dagger_{\mathrm{M}}$, $\alpha_{Y_{\mathrm{M}}}$ $\alpha^\dagger_{Y_{\mathrm{M}}}$, α_{χ^2}, α_{KP} (or α_{YKP}), α^*, and α_Y in each row of Tables 1-4 that are closest to 5 are in bold. We experimented with several other parameters, and found similar trends, which are not listed here for brevity.

For pairwise comparisons, random numbers were generated from a normal distribution as in the one- and two-sample problems. In Table 5, for $\alpha = 0.05$, $k = 3$, $(p_1, p_2, p_3) = (2, 2, 2)$, $n_i^{(1)} = n_i^{(2)} = \cdots = n_i^{(m)} = n_i, i = 1, 2, 3$, $m = 3, 6, 10$, $n_1 = 10(10)100, 200, 400$, $n_2 = n_1/2$, $n_3 = n_2$, the upper percentiles are denoted as follows.

i. $q_{\max}(\alpha)$ is the upper 100α percentile of $T^2_{\max}$ by simulation.

ii. $q_{\mathrm{Bon}}(\alpha_{\mathrm{p}})$ is the simulated upper $100\alpha_{\mathrm{p}}$ percentile of the $Q^{(ab)}$ statistic by simulation.

iii. $q_{\mathrm{MAE}}(\alpha_{\mathrm{p}})$ is given in (19).

iv. $q_{\mathrm{AE}}(\alpha_{\mathrm{p}})$ and $q_{\mathrm{YSL}}(\alpha_{\mathrm{p}})$ are approximate upper percentile points of $T^2_{\max}$ proposed in Yagi et al. (2018) and Yagi and Seo (2017), respectively.

In each row of Table 5, among $q_{\mathrm{MAE}}(\alpha_{\mathrm{p}})$, $q_{\mathrm{AE}}(\alpha_{\mathrm{p}})$, and $q_{\mathrm{YSL}}(\alpha_{\mathrm{p}})$, the conservative values and those closest to $q_{\max}(\alpha)$ values are shown in bold. When $m = 3$, the approximation accuracy of $q_{\mathrm{YSL}}(\alpha_{\mathrm{p}})$ was good when the sample size was quite small, and the approximation accuracy of $q_{\mathrm{AE}}(\alpha_{\mathrm{p}})$ was good for slightly larger sample sizes. It can be seen that the approximation accuracy of $q_{\mathrm{MAE}}(\alpha_{\mathrm{p}})$ was good for larger sample sizes. When $m = 6, 10$, the approximation accuracy of $q_{\mathrm{YSL}}(\alpha_{\mathrm{p}})$ and $q_{\mathrm{AE}}(\alpha_{\mathrm{p}})$ was good when the sample size was relatively small, and the approximation accuracy of $q_{\mathrm{MAE}}(\alpha_{\mathrm{p}})$ was good when the sample size was slightly larger. We also conducted numerical experiments for other parameters, but overall, we found the proposed approximate upper percentile $q_{\mathrm{MAE}}(\alpha_{\mathrm{p}})$ to be conservative, and it exhibited good approximation accuracy for reasonably large sample sizes.

Table 1: Upper percentiles of test statistics and empirical type I errors for a one-sample case with $(p_1, p_2, p_3) = (2, 2, 2)$ and $\alpha = 0.05$.

n_1	$n_2 = n_3$	q_{M} ($\alpha_{\mathrm{M}\chi^2}$)	q_{MAE} (α_{MAE})	q_{MKP} (α_{MKP})	q^*_{M} (α^*_{M})	$q^\dagger_{\mathrm{M}}$ ($\alpha^\dagger_{\mathrm{M}}$)	$q_{Y_{\mathrm{M}}}$ ($\alpha_{Y_{\mathrm{M}}}$)	$q^\dagger_{Y_{\mathrm{M}}}$ ($\alpha^\dagger_{Y_{\mathrm{M}}}$)	q (α_{χ^2})	q_{KP} (α_{KP})	q^* (α^*)	q_Y (α_Y)
15	7	21.81	17.26	22.49	13.63	14.95	12.57	13.69	29.26	29.37	12.36	11.86
		(22.38)	(10.26)	(4.52)	(6.60)	(8.54)	(**4.97**)	(6.91)	(31.52)	(4.95)	(4.70)	(3.87)
20	10	18.06	16.05	18.23	13.30	13.85	12.60	13.11	21.73	21.42	13.06	12.45
		(15.88)	(7.63)	(4.83)	(6.17)	(7.05)	(**5.02**)	(5.94)	(21.89)	(5.24)	(5.70)	(4.76)
30	15	15.62	14.90	15.67	13.00	13.19	12.58	12.78	17.36	17.08	13.09	12.66
		(11.09)	(6.06)	(4.94)	(5.71)	(6.04)	(**4.99**)	(5.34)	(14.39)	(5.32)	(5.81)	(5.12)
40	20	14.70	14.32	14.72	12.89	12.99	12.60	12.69	15.83	15.60	13.00	12.70
		(9.17)	(5.58)	(4.98)	(5.53)	(5.69)	(**5.01**)	(5.19)	(11.39)	(5.29)	(5.71)	(5.19)
50	25	14.21	13.98	14.21	12.83	12.89	12.60	12.66	15.03	14.85	12.94	12.69
		(8.15)	(5.37)	(**5.00**)	(5.42)	(5.52)	(5.01)	(5.12)	(9.78)	(5.25)	(5.60)	(5.18)
100	50	13.32	13.28	13.34	12.69	12.70	12.58	12.59	13.67	13.60	12.76	12.65
		(6.39)	(5.06)	(4.97)	(5.18)	(5.19)	(4.98)	(**5.00**)	(7.06)	(5.11)	(5.30)	(5.11)
200	100	12.95	12.94	12.95	12.65	12.65	12.60	12.60	13.12	13.07	12.69	12.63
		(5.68)	(5.03)	(**5.01**)	(5.11)	(5.11)	(**5.01**)	(**5.01**)	(6.00)	(5.08)	(5.18)	(5.08)
400	200	12.75	12.76	12.77	12.60	12.60	12.58	12.58	12.83	12.83	12.62	12.60
		(5.29)	(4.98)	(4.97)	(5.02)	(5.02)	(4.97)	(4.98)	(5.43)	(**5.01**)	(5.06)	(**5.01**)
800	400	12.68	12.68	12.68	12.60	12.60	12.59	12.59	12.71	12.71	12.61	12.60
		(5.15)	(**4.99**)	(**4.99**)	(5.02)	(5.02)	(**4.99**)	(**4.99**)	(5.22)	(**5.01**)	(5.03)	(**5.01**)
15	15	20.84	16.63	21.65	13.45	15.13	12.58	14.03	27.79	28.34	12.09	11.77
		(20.29)	(10.00)	(4.43)	(6.39)	(8.86)	(**4.98**)	(7.46)	(29.30)	(4.74)	(4.31)	(3.70)
20	20	17.47	15.62	17.69	13.19	13.88	12.60	13.25	20.79	20.79	12.83	12.33
		(14.58)	(7.49)	(4.77)	(5.99)	(7.12)	(5.01)	(6.16)	(20.33)	(**5.00**)	(5.37)	(4.57)
30	30	15.28	14.61	15.33	12.94	13.18	12.59	12.83	16.83	16.69	12.94	12.58
		(10.33)	(5.99)	(4.93)	(5.61)	(6.02)	(**4.99**)	(5.44)	(13.39)	(5.16)	(5.58)	(4.98)
40	40	14.46	14.11	14.47	12.85	12.97	12.59	12.72	15.46	15.32	12.90	12.64
		(8.61)	(5.54)	(4.98)	(5.45)	(5.66)	(**5.00**)	(5.24)	(10.63)	(5.19)	(5.53)	(5.08)
50	50	14.00	13.80	14.02	12.78	12.85	12.59	12.66	14.73	14.63	12.84	12.64
		(7.75)	(5.33)	(4.97)	(5.34)	(5.47)	(**4.99**)	(5.13)	(9.23)	(5.15)	(5.45)	(5.09)
100	100	13.24	13.20	13.25	12.67	12.69	12.58	12.60	13.55	13.50	12.72	12.62
		(6.21)	(5.06)	(4.98)	(5.15)	(5.18)	(4.97)	(**5.01**)	(6.81)	(5.08)	(5.23)	(5.05)
200	200	12.91	12.89	12.91	12.64	12.64	12.59	12.60	13.05	13.02	12.66	12.62
		(5.59)	(5.02)	(**5.00**)	(5.09)	(5.09)	(**5.00**)	(5.01)	(5.85)	(5.05)	(5.13)	(5.05)
400	400	12.74	12.74	12.75	12.60	12.61	12.58	12.59	12.80	12.80	12.62	12.60
		(5.27)	(**4.99**)	(4.98)	(5.03)	(5.03)	(4.98)	(**4.99**)	(5.40)	(**5.01**)	(5.05)	(**5.01**)
800	800	12.68	12.67	12.67	12.61	12.61	12.60	12.60	12.71	12.69	12.62	12.61
		(5.16)	(**5.02**)	(**5.02**)	(5.04)	(5.04)	(**5.02**)	(**5.02**)	(5.22)	(5.03)	(5.05)	(5.03)
15	30	20.26	16.11	21.06	13.35	15.40	12.59	14.41	26.88	27.61	11.87	11.67
		(18.74)	(9.95)	(4.42)	(6.20)	(9.18)	(**4.99**)	(8.02)	(27.55)	(4.66)	(3.98)	(3.55)
20	40	17.00	15.23	17.25	13.09	13.94	12.59	13.40	20.11	20.28	12.65	12.24
		(13.56)	(7.44)	(4.73)	(5.85)	(7.23)	(**5.00**)	(6.44)	(18.99)	(4.86)	(5.09)	(4.41)
30	60	14.97	14.35	15.05	12.89	13.18	12.60	12.88	16.38	16.35	12.83	12.54
		(9.67)	(5.96)	(4.90)	(5.51)	(6.03)	(**5.01**)	(5.54)	(12.48)	(5.04)	(5.40)	(4.90)
40	80	14.23	13.91	14.26	12.79	12.95	12.58	12.74	15.12	15.07	12.80	12.57
		(8.19)	(5.52)	(4.95)	(5.36)	(5.64)	(**4.98**)	(5.29)	(9.98)	(5.07)	(5.35)	(4.97)
50	100	13.85	13.65	13.86	12.76	12.85	12.60	12.69	14.50	14.44	12.78	12.61
		(7.44)	(5.33)	(**4.99**)	(5.30)	(5.47)	(5.03)	(5.18)	(8.74)	(5.10)	(5.33)	(5.03)
100	200	13.14	13.12	13.17	12.64	12.66	12.57	12.59	13.40	13.40	12.67	12.59
		(6.03)	(5.03)	(4.95)	(5.09)	(5.13)	(4.95)	(4.99)	(6.56)	(**5.00**)	(5.14)	(**5.00**)
200	400	12.86	12.86	12.87	12.62	12.63	12.58	12.59	12.98	12.97	12.63	12.60
		(5.50)	(**5.00**)	(4.98)	(5.05)	(5.07)	(4.98)	(**5.00**)	(5.74)	(5.01)	(5.08)	(5.01)
400	800	12.72	12.72	12.73	12.61	12.61	12.59	12.59	12.79	12.78	12.62	12.60
		(5.24)	(**5.00**)	(**5.00**)	(5.03)	(5.04)	(**5.00**)	(**5.00**)	(5.35)	(5.01)	(5.04)	(5.01)
800	1600	12.66	12.66	12.66	12.60	12.60	12.59	12.59	12.69	12.68	12.61	12.60
		(5.12)	(5.01)	(5.01)	(5.02)	(5.02)	(**5.00**)	(5.01)	(5.17)	(5.01)	(5.03)	(5.01)

Note. $\chi^2_6(0.05) = 12.59$.

Table 2: Upper percentiles of test statistics and empirical type I errors for a one-sample case with $(p_1, p_2, p_3, p_4, p_5) = (2, 2, 2, 2, 2)$ and $\alpha = 0.05$.

n_1	$n_2 = \cdots = n_5$	q_{M}	q_{MAE}	q_{MKP}	q^*_{M}	$q^\dagger_{\mathrm{M}}$	$q_{\mathrm{Y_M}}$	$q^\dagger_{\mathrm{Y_M}}$	q	q_{KP}	q^*	q_Y
		$(\alpha_{\mathrm{M}\chi^2})$	(α_{MAE})	(α_{MKP})	(α^*_{M})	$(\alpha^\dagger_{\mathrm{M}})$	$(\alpha_{\mathrm{Y_M}})$	$(\alpha^\dagger_{\mathrm{Y_M}})$	(α_{χ^2})	(α_{KP})	(α^*)	(α_Y)
18	9	33.59	24.30	35.23	19.42	23.74	18.33	22.12	57.29	58.40	14.70	15.44
		(30.80)	(14.69)	(4.19)	(6.53)	(12.07)	(**5.03**)	(10.53)	(50.48)	(4.75)	(1.90)	(2.06)
20	10	29.92	23.70	30.74	19.22	22.02	18.32	20.87	45.70	45.93	16.06	16.37
		(25.91)	(12.03)	(4.48)	(6.31)	(10.15)	(**5.02**)	(8.87)	(43.11)	(4.92)	(2.72)	(2.78)
30	15	23.79	21.90	23.84	18.82	19.60	18.33	19.09	29.17	28.50	18.41	17.99
		(15.18)	(7.39)	(4.95)	(5.77)	(6.93)	(**5.04**)	(6.21)	(24.41)	(5.52)	(5.12)	(4.57)
40	20	21.87	21.00	21.92	18.63	18.98	18.28	18.65	24.99	24.52	18.63	18.30
		(11.48)	(6.14)	(4.93)	(5.47)	(6.02)	(**4.97**)	(5.52)	(17.26)	(5.46)	(5.44)	(4.98)
50	25	21.00	20.46	21.00	18.57	18.78	18.31	18.52	23.18	22.78	18.68	18.41
		(9.75)	(5.73)	(**4.99**)	(5.39)	(5.73)	(**5.01**)	(5.34)	(13.81)	(5.44)	(5.52)	(5.15)
100	50	19.49	19.39	19.50	18.42	18.46	18.30	18.34	20.35	20.18	18.54	18.41
		(6.97)	(5.15)	(**4.98**)	(5.16)	(5.23)	(**4.98**)	(5.05)	(8.49)	(5.23)	(5.34)	(5.15)
200	100	18.87	18.85	18.87	18.36	18.37	18.30	18.31	19.25	19.18	18.43	18.37
		(5.91)	(5.03)	(**5.00**)	(5.08)	(5.10)	(4.99)	(5.01)	(6.56)	(5.12)	(5.19)	(5.11)
400	200	18.59	18.58	18.58	18.35	18.35	18.31	18.32	18.78	18.73	18.38	18.35
		(5.45)	(5.03)	(5.02)	(5.06)	(5.07)	(**5.01**)	(5.02)	(5.75)	(5.08)	(5.12)	(5.07)
800	400	18.43	18.44	18.44	18.31	18.31	18.29	18.29	18.52	18.51	18.33	18.31
		(5.20)	(4.98)	(4.98)	(**5.00**)	(**5.00**)	(4.98)	(4.98)	(5.34)	(5.01)	(5.03)	(5.01)
18	18	31.90	23.26	33.73	19.19	24.11	18.31	22.78	53.84	55.72	14.77	15.57
		(26.93)	(14.14)	(4.10)	(6.24)	(12.50)	(**5.00**)	(11.29)	(45.93)	(4.59)	(1.80)	(2.10)
20	20	28.53	22.77	29.42	19.02	22.27	18.31	21.33	42.86	43.86	15.93	16.36
		(22.63)	(11.55)	(4.41)	(6.05)	(10.41)	(**5.00**)	(9.40)	(38.96)	(4.67)	(2.50)	(2.71)
30	30	22.92	21.28	23.05	18.69	19.56	18.29	19.17	27.54	27.34	18.16	17.86
		(13.39)	(7.14)	(4.86)	(5.56)	(6.88)	(**4.98**)	(6.32)	(21.47)	(5.16)	(4.80)	(4.38)
40	40	21.32	20.54	21.36	18.57	18.98	18.30	18.72	23.95	23.70	18.45	18.19
		(10.37)	(6.08)	(4.95)	(5.39)	(6.03)	(**4.99**)	(5.64)	(15.27)	(5.26)	(5.21)	(4.83)
50	50	20.56	20.09	20.57	18.53	18.76	18.32	18.56	22.42	22.16	18.54	18.33
		(8.92)	(5.66)	(**4.99**)	(5.34)	(5.69)	(5.02)	(5.39)	(12.31)	(5.30)	(5.35)	(5.03)
100	100	19.31	19.20	19.30	18.41	18.46	18.31	18.37	20.01	19.90	18.47	18.37
		(6.66)	(5.16)	(**5.01**)	(5.16)	(5.23)	(**5.01**)	(5.09)	(7.91)	(5.16)	(5.26)	(5.11)
200	200	18.76	18.75	18.78	18.34	18.35	18.30	18.30	19.08	19.04	18.39	18.34
		(5.73)	(5.01)	(4.98)	(5.05)	(5.06)	(4.98)	(**5.00**)	(6.25)	(5.06)	(5.12)	(5.05)
400	400	18.53	18.53	18.54	18.33	18.33	18.31	18.31	18.68	18.66	18.35	18.33
		(5.36)	(5.01)	(**5.00**)	(5.03)	(5.04)	(**5.00**)	(**5.00**)	(5.60)	(5.04)	(5.07)	(5.03)
800	800	18.41	18.42	18.42	18.31	18.31	18.29	18.29	18.48	18.48	18.32	18.31
		(5.15)	(4.98)	(4.98)	(**5.00**)	(**5.00**)	(4.98)	(4.98)	(5.27)	(**5.00**)	(5.02)	(**5.00**)
18	36	30.74	22.40	32.70	19.03	24.51	18.29	23.42	51.73	53.87	14.72	15.59
		(24.22)	(13.89)	(4.05)	(6.01)	(12.88)	(**4.98**)	(11.94)	(42.58)	(4.53)	(1.63)	(2.05)
20	40	27.55	21.99	28.51	18.90	22.52	18.28	21.77	41.16	42.43	15.77	16.32
		(20.21)	(11.41)	(4.36)	(5.85)	(10.74)	(**4.97**)	(9.95)	(35.58)	(4.58)	(2.27)	(2.62)
30	60	22.34	20.76	22.45	18.63	19.62	18.32	19.31	26.45	26.48	18.00	17.80
		(12.10)	(7.08)	(4.88)	(5.50)	(6.95)	(**5.02**)	(6.52)	(19.24)	(**4.98**)	(4.58)	(4.27)
40	80	20.89	20.15	20.93	18.54	18.99	18.33	18.78	23.18	23.07	18.34	18.14
		(9.49)	(6.04)	(4.96)	(5.35)	(6.05)	(**5.03**)	(5.75)	(13.71)	(5.12)	(5.06)	(4.75)
50	100	20.18	19.78	20.23	18.44	18.71	18.28	18.55	21.73	21.66	18.37	18.21
		(8.20)	(5.56)	(4.93)	(5.20)	(5.62)	(**4.96**)	(5.38)	(11.07)	(5.08)	(5.10)	(4.85)
100	200	19.17	19.04	19.14	18.41	18.47	18.33	18.40	19.76	19.66	18.44	18.35
		(6.39)	(5.19)	(5.05)	(5.16)	(5.26)	(**5.04**)	(5.14)	(7.40)	(5.14)	(5.20)	(5.07)
200	400	18.70	18.68	18.70	18.34	18.36	18.31	18.32	18.97	18.93	18.37	18.33
		(5.63)	(5.04)	(**5.00**)	(5.06)	(5.08)	(**5.00**)	(5.02)	(6.07)	(5.05)	(5.09)	(5.04)
400	800	18.49	18.49	18.50	18.32	18.32	18.30	18.31	18.62	18.60	18.34	18.32
		(5.29)	(**5.00**)	(**5.00**)	(5.02)	(5.03)	(4.99)	(**5.00**)	(5.49)	(5.02)	(5.04)	(5.02)
800	1600	18.42	18.40	18.40	18.33	18.33	18.32	18.32	18.48	18.45	18.34	18.33
		(5.17)	(5.03)	(5.03)	(5.04)	(5.04)	(**5.02**)	(5.03)	(5.25)	(5.03)	(5.04)	(5.03)

Note. $\chi^2_{10}(0.05) = 18.31$.

Table 3: Upper percentiles of test statistics and empirical type I errors for a two-sample case with $(p_1, p_2, p_3) = (2, 2, 2)$ and $\alpha = 0.05$.

$n_1^{(\ell)}$	$n_2^{(\ell)} = n_3^{(\ell)}$	q_{M} ($\alpha_{\mathrm{M}\chi^2}$)	q_{MAE} (α_{MAE})	q_{MYKP} (α_{MYKP})	q_{M}^* (α_{M}^*)	$q_{\mathrm{M}}^\dagger$ ($\alpha_{\mathrm{M}}^\dagger$)	$q_{Y_{\mathrm{M}}}$ ($\alpha_{Y_{\mathrm{M}}}$)	$q_{Y_{\mathrm{M}}}^\dagger$ ($\alpha_{Y_{\mathrm{M}}}^\dagger$)	q (α_{χ^2})	q_{YKP} (α_{YKP})	q^* (α^*)	q_Y (α_Y)
10	5	19.25	16.51	19.48	13.32	14.06	12.59	13.26	23.55	23.27	12.99	12.37
		(18.41)	(8.48)	(4.78)	(6.19)	(7.35)	(**4.99**)	(6.19)	(25.05)	(5.19)	(5.57)	(4.63)
20	10	15.04	14.55	15.06	12.90	13.01	12.59	12.71	16.24	16.00	13.01	12.70
		(9.94)	(5.75)	(4.97)	(5.54)	(5.75)	(**5.01**)	(5.22)	(12.28)	(5.30)	(5.70)	(5.19)
30	15	14.10	13.90	14.10	12.79	12.84	12.60	12.65	14.79	14.62	12.89	12.69
		(7.99)	(5.34)	(**5.00**)	(5.37)	(5.44)	(5.02)	(5.10)	(9.31)	(5.25)	(5.52)	(5.18)
40	20	13.68	13.57	13.68	12.73	12.76	12.59	12.62	14.15	14.04	12.82	12.67
		(7.10)	(5.17)	(**5.00**)	(5.25)	(5.29)	(**5.00**)	(5.05)	(8.02)	(5.17)	(5.39)	(5.14)
50	25	13.44	13.37	13.44	12.70	12.72	12.59	12.61	13.80	13.71	12.77	12.66
		(6.62)	(5.11)	(**4.99**)	(5.19)	(5.22)	(**4.99**)	(5.03)	(7.32)	(5.14)	(5.32)	(5.12)
100	50	12.99	12.98	13.00	12.63	12.64	12.58	12.58	13.15	13.12	12.67	12.61
		(5.74)	(**5.01**)	(4.98)	(5.08)	(5.08)	(4.97)	(**4.99**)	(6.05)	(5.04)	(5.14)	(5.04)
200	100	12.79	12.79	12.79	12.61	12.61	12.59	12.59	12.87	12.85	12.63	12.61
		(5.36)	(**5.00**)	(4.99)	(5.04)	(5.04)	(4.99)	(4.99)	(5.51)	(5.03)	(5.08)	(5.03)
400	200	12.70	12.69	12.69	12.61	12.61	12.60	12.60	12.74	12.72	12.62	12.61
		(5.19)	(**5.01**)	(**5.01**)	(5.03)	(5.03)	(**5.01**)	(**5.01**)	(5.27)	(5.03)	(5.05)	(5.03)
800	400	12.63	12.64	12.64	12.59	12.59	12.58	12.58	12.65	12.65	12.60	12.59
		(5.07)	(4.99)	(4.99)	(**5.00**)	(**5.00**)	(4.99)	(4.99)	(5.11)	(**5.00**)	(5.01)	(4.99)
10	10	18.53	16.01	18.82	13.22	14.16	12.60	13.46	22.45	22.51	12.77	12.27
		(16.74)	(8.30)	(4.72)	(6.05)	(7.53)	(**5.02**)	(6.52)	(23.12)	(4.96)	(5.28)	(4.46)
20	20	14.73	14.30	14.77	12.83	12.99	12.58	12.73	15.79	15.67	12.88	12.62
		(9.28)	(5.67)	(4.94)	(5.42)	(5.71)	(**4.98**)	(5.26)	(11.41)	(5.17)	(5.50)	(5.05)
30	30	13.90	13.73	13.92	12.74	12.80	12.58	12.64	14.49	14.41	12.80	12.64
		(7.56)	(5.27)	(4.97)	(5.28)	(5.39)	(**4.98**)	(5.09)	(8.75)	(5.12)	(5.38)	(5.08)
40	40	13.55	13.45	13.55	12.72	12.75	12.60	12.64	13.97	13.88	12.78	12.66
		(6.84)	(5.18)	(**5.01**)	(5.23)	(5.29)	(5.02)	(5.08)	(7.67)	(5.13)	(5.33)	(5.12)
50	50	13.33	13.27	13.34	12.68	12.70	12.59	12.61	13.64	13.59	12.73	12.63
		(6.40)	(5.09)	(**4.99**)	(5.16)	(5.20)	(**4.99**)	(5.03)	(7.02)	(5.08)	(5.25)	(5.07)
100	100	12.92	12.93	12.95	12.62	12.62	12.57	12.57	13.07	13.06	12.64	12.60
		(5.62)	(4.98)	(4.96)	(5.04)	(5.04)	(4.96)	(4.97)	(5.90)	(**5.00**)	(5.09)	(5.01)
200	200	12.75	12.76	12.77	12.60	12.60	12.58	12.58	12.82	12.82	12.61	12.59
		(5.31)	(4.98)	(4.98)	(5.02)	(5.02)	(4.98)	(4.98)	(5.44)	(**5.00**)	(5.04)	(**5.00**)
400	400	12.71	12.68	12.68	12.63	12.63	12.62	12.62	12.74	12.70	12.64	12.63
		(5.22)	(5.06)	(**5.05**)	(5.08)	(5.08)	(**5.05**)	(5.06)	(5.28)	(5.07)	(5.09)	(5.07)
800	800	12.63	12.63	12.63	12.59	12.59	12.58	12.58	12.64	12.65	12.59	12.59
		(5.07)	(4.98)	(4.98)	(4.99)	(**5.00**)	(4.98)	(4.98)	(5.10)	(4.99)	(**5.00**)	(4.99)
10	20	17.91	15.55	18.30	13.07	14.23	12.57	13.64	21.60	21.92	12.56	12.16
		(15.41)	(8.20)	(4.62)	(5.82)	(7.69)	(**4.96**)	(6.85)	(21.54)	(4.77)	(4.95)	(4.27)
20	40	14.49	14.07	14.52	12.82	13.00	12.60	12.79	15.44	15.38	12.81	12.60
		(8.72)	(5.67)	(4.96)	(5.40)	(5.73)	(5.02)	(5.37)	(10.64)	(5.08)	(5.39)	(**5.01**)
30	60	13.75	13.58	13.76	12.74	12.81	12.61	12.68	14.28	14.22	12.77	12.63
		(7.24)	(5.29)	(**5.00**)	(5.27)	(5.40)	(5.03)	(5.16)	(8.31)	(5.09)	(5.31)	(5.06)
40	80	13.42	13.33	13.43	12.69	12.73	12.59	12.64	13.79	13.74	12.72	12.61
		(6.57)	(5.15)	(**5.00**)	(5.18)	(5.25)	(4.99)	(5.08)	(7.28)	(5.08)	(5.23)	(5.04)
50	100	13.25	13.18	13.24	12.68	12.71	12.60	12.63	13.53	13.48	12.71	12.62
		(6.24)	(5.12)	(**5.01**)	(5.16)	(5.21)	(**5.01**)	(5.07)	(6.77)	(5.08)	(5.20)	(5.06)
100	200	12.89	12.89	12.90	12.62	12.63	12.58	12.59	13.02	13.01	12.63	12.60
		(5.55)	(5.01)	(4.98)	(5.05)	(5.06)	(4.99)	(**5.00**)	(5.79)	(5.02)	(5.08)	(5.01)
200	400	12.74	12.74	12.74	12.61	12.61	12.59	12.59	12.80	12.79	12.62	12.60
		(5.26)	(**5.00**)	(**5.00**)	(5.03)	(5.03)	(**5.00**)	(**5.00**)	(5.38)	(5.01)	(5.04)	(5.01)
400	800	12.65	12.67	12.67	12.59	12.59	12.58	12.58	12.68	12.69	12.59	12.58
		(5.11)	(4.98)	(4.97)	(4.99)	(4.99)	(4.97)	(4.98)	(5.16)	(4.98)	(**5.00**)	(4.98)
800	1600	12.63	12.63	12.63	12.60	12.60	12.60	12.60	12.65	12.64	12.60	12.60
		(5.08)	(**5.01**)	(**5.01**)	(5.02)	(**5.01**)	(**5.01**)	(**5.01**)	(5.10)	(**5.01**)	(5.02)	(**5.01**)

Note. $\chi_6^2(0.05) = 12.59$.

Table 4: Upper percentiles of test statistics and empirical type I errors for a two-sample case with $(p_1, p_2, p_3, p_4, p_5) = (2, 2, 2, 2, 2)$ and $\alpha = 0.05$.

$n_1^{(\ell)}$	$n_2^{(\ell)} = \cdots = n_5^{(\ell)}$	q_{M} ($\alpha_{\mathrm{M}\chi^2}$)	q_{MAE} (α_{MAE})	q_{MYKP} (α_{MYKP})	q_{M}^* (α_{M}^*)	$q_{\mathrm{M}}^\dagger$ ($\alpha_{\mathrm{M}}^\dagger$)	$q_{Y_{\mathrm{M}}}$ ($\alpha_{Y_{\mathrm{M}}}$)	$q_{Y_{\mathrm{M}}}^\dagger$ ($\alpha_{Y_{\mathrm{M}}}^\dagger$)	q (α_{χ^2})	q_{YKP} (α_{YKP})	q^* (α^*)	q_Y (α_Y)
10	5	32.62	24.23	33.78	19.31	23.06	18.33	21.69	52.53	53.13	15.41	15.95
		(30.08)	(14.08)	(4.37)	(6.41)	(11.39)	(**5.03**)	(9.98)	(48.40)	(4.84)	(2.29)	(2.43)
20	10	22.32	21.27	22.39	18.62	19.05	18.28	18.70	25.65	25.17	18.61	18.26
		(12.41)	(6.40)	(4.92)	(5.48)	(6.14)	(**4.95**)	(5.62)	(18.53)	(5.46)	(5.41)	(4.93)
30	15	20.70	20.28	20.71	18.52	18.68	18.31	18.47	22.43	22.11	18.65	18.43
		(9.23)	(5.57)	(4.99)	(5.34)	(5.56)	(**5.00**)	(5.24)	(12.46)	(5.38)	(5.51)	(5.19)
40	20	20.02	19.79	20.01	18.46	18.55	18.31	18.40	21.19	20.94	18.60	18.44
		(7.90)	(5.32)	(**5.01**)	(5.25)	(5.37)	(**5.01**)	(5.14)	(10.04)	(5.30)	(5.44)	(5.20)
50	25	19.64	19.49	19.63	18.44	18.49	18.32	18.37	20.51	20.32	18.56	18.44
		(7.23)	(5.21)	(**5.02**)	(5.21)	(5.28)	(5.03)	(5.10)	(8.79)	(5.26)	(5.38)	(5.19)
100	50	18.91	18.90	18.93	18.34	18.35	18.28	18.30	19.30	19.24	18.41	18.35
		(5.96)	(**5.01**)	(4.96)	(5.05)	(5.07)	(4.96)	(4.98)	(6.62)	(5.08)	(5.15)	(5.06)
200	100	18.62	18.60	18.61	18.35	18.35	18.32	18.32	18.81	18.75	18.39	18.36
		(5.50)	(5.03)	(**5.02**)	(5.06)	(5.07)	(**5.02**)	(**5.02**)	(5.80)	(5.09)	(5.12)	(5.08)
400	200	18.47	18.46	18.46	18.34	18.34	18.32	18.32	18.56	18.53	18.35	18.34
		(5.25)	(5.03)	(**5.02**)	(5.04)	(5.05)	(**5.02**)	(**5.02**)	(5.40)	(5.05)	(5.07)	(5.05)
800	400	18.38	18.38	18.38	18.31	18.31	18.30	18.30	18.42	18.42	18.32	18.31
		(5.12)	(**5.00**)	(**5.00**)	(5.01)	(5.01)	(4.99)	(**5.00**)	(5.18)	(5.01)	(5.02)	(5.01)
10	10	30.92	23.19	32.26	19.12	23.42	18.33	22.30	49.37	50.66	15.39	16.02
		(26.24)	(13.51)	(4.27)	(6.15)	(11.84)	(**5.03**)	(10.75)	(43.94)	(4.66)	(2.14)	(2.43)
20	20	21.72	20.75	21.75	18.59	19.09	18.32	18.82	24.55	24.26	18.47	18.21
		(11.17)	(6.30)	(**4.98**)	(5.44)	(6.18)	(**5.02**)	(5.79)	(16.44)	(5.30)	(5.24)	(4.84)
30	30	20.31	19.93	20.31	18.49	18.67	18.32	18.50	21.75	21.55	18.53	18.36
		(8.45)	(5.53)	(**5.00**)	(5.28)	(5.55)	(5.02)	(5.30)	(11.17)	(5.24)	(5.33)	(5.07)
40	40	19.72	19.53	19.72	18.43	18.53	18.31	18.41	20.70	20.54	18.50	18.37
		(7.38)	(5.28)	(**5.00**)	(5.19)	(5.33)	(**5.00**)	(5.16)	(9.11)	(5.20)	(5.29)	(5.10)
50	50	19.42	19.28	19.40	18.42	18.48	18.33	18.39	20.14	20.01	18.49	18.39
		(6.84)	(5.21)	(**5.03**)	(5.18)	(5.27)	(**5.03**)	(5.13)	(8.12)	(5.18)	(5.28)	(5.12)
100	100	18.83	18.79	18.82	18.36	18.38	18.32	18.33	19.16	19.09	18.41	18.36
		(5.84)	(5.06)	(**5.02**)	(5.09)	(5.12)	(**5.02**)	(5.04)	(6.36)	(5.10)	(5.16)	(5.09)
200	200	18.59	18.55	18.56	18.36	18.36	18.34	18.34	18.74	18.68	18.38	18.36
		(5.44)	(5.05)	(**5.04**)	(5.08)	(5.08)	(5.05)	(5.05)	(5.68)	(5.08)	(5.11)	(5.08)
400	400	18.43	18.43	18.43	18.32	18.32	18.31	18.31	18.51	18.49	18.33	18.32
		(5.20)	(5.01)	(**5.00**)	(5.02)	(5.02)	(**5.00**)	(5.01)	(5.32)	(5.02)	(5.04)	(5.02)
800	800	18.35	18.37	18.37	18.30	18.30	18.29	18.29	18.39	18.40	18.30	18.30
		(5.07)	(4.98)	(4.98)	(4.98)	(4.98)	(4.98)	(4.98)	(5.13)	(**4.99**)	(**4.99**)	(4.98)
10	20	29.82	22.32	31.22	18.96	23.85	18.31	22.93	47.53	48.97	15.30	16.01
		(23.46)	(13.32)	(4.23)	(5.96)	(12.31)	(**5.00**)	(11.46)	(40.43)	(4.62)	(1.95)	(2.37)
20	40	21.21	20.31	21.25	18.54	19.09	18.32	18.88	23.63	23.57	18.31	18.12
		(10.07)	(6.23)	(4.95)	(5.35)	(6.18)	(5.02)	(5.87)	(14.62)	(5.07)	(**5.01**)	(4.71)
30	60	20.01	19.64	19.99	18.47	18.67	18.34	18.55	21.24	21.11	18.45	18.31
		(7.90)	(5.52)	(5.02)	(5.25)	(5.57)	(5.04)	(5.37)	(10.13)	(5.17)	(5.21)	(**5.01**)
40	80	19.50	19.31	19.49	18.43	18.53	18.33	18.43	20.32	20.22	18.44	18.34
		(6.99)	(5.29)	(**5.02**)	(5.18)	(5.35)	(5.03)	(5.20)	(8.43)	(5.14)	(5.21)	(5.05)
50	100	19.21	19.11	19.22	18.38	18.44	18.30	18.37	19.81	19.76	18.40	18.33
		(6.47)	(5.15)	(4.98)	(5.11)	(5.21)	(**4.99**)	(5.10)	(7.54)	(5.08)	(5.15)	(5.03)
100	200	18.75	18.71	18.73	18.36	18.37	18.32	18.34	19.01	18.97	18.38	18.35
		(5.71)	(5.07)	(5.03)	(5.08)	(5.11)	(**5.02**)	(5.05)	(6.14)	(5.07)	(5.12)	(5.06)
200	400	18.50	18.51	18.51	18.31	18.31	18.29	18.29	18.62	18.62	18.32	18.31
		(5.30)	(4.98)	(4.97)	(**5.00**)	(5.01)	(4.97)	(4.98)	(5.49)	(**5.00**)	(5.03)	(**5.00**)
400	800	18.38	18.41	18.41	18.29	18.29	18.28	18.28	18.44	18.46	18.30	18.29
		(5.12)	(4.96)	(4.96)	(4.97)	(4.97)	(4.96)	(4.96)	(5.21)	(4.97)	(**4.98**)	(4.97)
800	1600	18.37	18.36	18.36	18.32	18.32	18.32	18.32	18.40	18.38	18.33	18.32
		(5.09)	(**5.02**)	(**5.02**)	(**5.02**)	(**5.02**)	(**5.02**)	(**5.02**)	(5.14)	(**5.02**)	(5.03)	(**5.02**)

Note. $\chi^2_{10}(0.05) = 18.31$.

Table 5: Simulated and approximate values for pairwise comparisons with $(p_1, p_2, p_3) = (2, 2, 2)$ and $\alpha = 0.05$.

n_1	$n_2 = n_3$	$q_{\max}(\alpha)$	$q_{\mathrm{Bon}}(\alpha)$	$q_{\mathrm{MAE}}(\alpha_{\mathrm{p}})$	$q_{\mathrm{AE}}(\alpha_{\mathrm{p}})$	$q_{\mathrm{YSL}}(\alpha_{\mathrm{p}})$	$\chi_p^2(\alpha_{\mathrm{p}})$
				$m = 3$			
10	5	23.77	24.48	19.36	20.28	23.74	15.51
20	10	18.47	18.88	17.44	17.89	**18.74**	15.51
30	15	17.21	17.56	16.79	17.10	**17.52**	15.51
40	20	16.67	16.99	16.47	**16.70**	16.97	15.51
50	25	16.36	16.67	16.28	**16.46**	16.65	15.51
60	30	16.17	16.46	16.15	**16.30**	16.45	15.51
70	35	16.05	16.35	**16.06**	16.19	16.31	15.51
80	40	15.92	16.19	**15.99**	16.10	16.20	15.51
90	45	15.83	16.08	**15.93**	16.04	16.12	15.51
100	50	15.78	16.08	**15.89**	15.98	16.06	15.51
200	100	15.50	15.78	15.70	15.74	15.78	**15.51**
400	200	15.38	15.63	15.60	15.63	15.64	**15.51**
800	400	15.31	15.54	15.55	15.57	15.57	**15.51**
				$m = 6$			
10	5	24.90	25.89	22.92	23.50	**25.60**	19.55
20	10	21.56	22.12	21.23	21.52	**22.15**	19.55
30	15	20.67	21.21	**20.67**	20.87	21.21	19.55
40	20	20.23	20.75	**20.39**	20.54	20.77	19.55
50	25	20.02	20.54	**20.22**	20.34	20.51	19.55
60	30	19.85	20.33	**20.11**	20.21	20.34	19.55
70	35	19.72	20.17	**20.03**	20.11	20.23	19.55
80	40	19.68	20.10	**19.97**	20.04	20.14	19.55
90	45	19.59	20.03	**19.92**	19.99	20.07	19.55
100	50	19.55	20.06	19.88	19.94	20.02	**19.55**
200	100	19.33	19.82	19.72	19.74	19.78	**19.55**
400	200	19.20	19.58	19.63	19.65	19.66	**19.55**
800	400	19.18	19.59	19.59	19.60	19.60	**19.55**
				$m = 10$			
10	5	26.31	27.12	25.26	25.65	**27.16**	22.21
20	10	23.76	24.33	23.73	**23.93**	24.42	22.21
30	15	23.02	23.58	**23.22**	23.35	23.63	22.21
40	20	22.66	23.26	**22.97**	23.07	23.26	22.21
50	25	22.48	23.02	**22.82**	22.89	23.04	22.21
60	30	22.33	22.87	**22.71**	22.78	22.90	22.21
70	35	22.24	22.71	**22.64**	22.70	22.80	22.21
80	40	22.18	22.70	22.59	22.64	22.72	**22.21**
90	45	22.15	22.63	22.54	22.59	22.66	**22.21**
100	50	22.07	22.58	22.51	22.55	22.62	**22.21**
200	100	21.90	22.37	22.36	22.38	22.41	**22.21**
400	200	21.81	22.40	22.28	22.29	22.31	**22.21**
800	400	21.75	22.30	22.24	22.25	22.26	**22.21**

Note. $\alpha_{\mathrm{p}} = 2 \cdot 0.05/\{m(m-1)\}$.

6 Power comparison

In this section, we show the results of a numerical power comparison of the test using statistic (i) $Q_{\mathrm{M}(\ell,k)}$, (ii) $Y_{\mathrm{M}(\ell,k)}$, (iii) $Q_{(\ell,k)}$, and (iv) $Y_{(\ell,k)}$ with three-step monotone missing data pattern for one-sample and two-sample cases ($k = 3; \ell = 1, 2$). The powers of (i), (ii), (iii), and (iv) were compared using corresponding simulated upper 100α percentiles under the null distribution for some parameter settings. As in the simulation detailed in the previous section, the simulation was executed 10^6 times using standard normal random vectors. The powers were computed with various values of $\delta_i = \boldsymbol{\mu}_i'\boldsymbol{\mu}_i$ $(i = 1, 2, 3)$ for the one-sample problem and $\delta_i = (\boldsymbol{\mu}_i^{(1)} - \boldsymbol{\mu}_i^{(2)})'(\boldsymbol{\mu}_i^{(1)} - \boldsymbol{\mu}_i^{(2)})$ $(i = 1, 2, 3)$ for the two-sample problem. Simulations for power computation of $Q_{(2,2)}$ and $Y_{(2,2)}$ were conducted by Yagi et al. (2023), and we also experimented in the same manner. Tables 6 and 7 show the power of four tests for one- and two-sample cases, respectively, for $(p_1, p_2, p_3) = (2, 2, 2)$ and $\alpha = 0.05$. From Table 6, it may be observed that the power of $Y_{(1,3)}$ was high when $\delta_2 = \delta_3 = 0$. Further, when $\delta_1 = \delta_3 = 0$, it may be observed that the power of $Y_{(1,3)}$ was high when the value of δ_2 was small, and the power of $Y_{\mathrm{M}(1,3)}$ was high when the value of δ_2 was large. Furthermore, when $\delta_1 = \delta_2 = 0$, it may be noted that the powers of $Q_{\mathrm{M}(1,3)}$ and $Q_{(1,3)}$ were almost equal and high. The results for the two-sample problem are shown in Table 7, and the same tendency as that of the one-sample problem was observed.

Table 6: Power comparison of (i), (ii), (iii), and (iv) for the one-sample problem.

δ_1	δ_2	δ_3	(i)	(ii)	(iii)	(iv)
			$(n_1, n_2, n_3) = (20, 10, 10)$			
0.0100	0	0	0.0588	0.0644	0.0582	**0.0659**
0.0625	0	0	0.1245	0.1610	0.1161	**0.1717**
0.2500	0	0	0.4774	0.5800	0.4266	**0.6041**
0.5625	0	0	0.8900	0.9357	0.8475	**0.9435**
1.0000	0	0	0.9948	0.9980	0.9895	**0.9983**
1.5625	0	0	**1.0000**	**1.0000**	0.9999	**1.0000**
2.2500	0	0	**1.0000**	**1.0000**	**1.0000**	**1.0000**
0	0.0100	0	0.0579	0.0595	0.0577	**0.0600**
0	0.0625	0	0.1079	0.1181	0.1039	**0.1204**
0	0.2500	0	0.3662	0.3964	0.3382	**0.3998**
0	0.5625	0	0.7599	**0.7825**	0.7195	0.7814
0	1.0000	0	0.9640	**0.9687**	0.9493	0.9678
0	1.5625	0	0.9981	**0.9984**	0.9966	0.9983
0	2.2500	0	**1.0000**	**1.0000**	0.9999	**1.0000**
0	3.0625	0	**1.0000**	**1.0000**	**1.0000**	**1.0000**
0	0	0.0100	**0.0570**	0.0546	**0.0570**	0.0536
0	0	0.0625	**0.0957**	0.0808	0.0947	0.0721
0	0	0.2500	**0.2699**	0.2019	0.2647	0.1575
0	0	0.5625	**0.5642**	0.4368	0.5568	0.3320
0	0	1.0000	**0.8297**	0.7094	0.8271	0.5742
0	0	1.5625	0.9584	0.8973	**0.9600**	0.7976
0	0	2.2500	0.9934	0.9755	**0.9947**	0.9316
0	0	3.0625	0.9992	0.9959	**0.9996**	0.9841
0	0	4.0000	0.9999	0.9995	**1.0000**	0.9975
0	0	5.0625	**1.0000**	0.9999	**1.0000**	0.9997
			$(n_1, n_2, n_3) = (30, 15, 15)$			
0.0100	0	0	0.0687	0.0734	0.0673	**0.0747**
0.0625	0	0	0.2049	0.2366	0.1932	**0.2444**
0.2500	0	0	0.7628	0.8056	0.7327	**0.8128**
0.5625	0	0	0.9919	0.9948	0.9886	**0.9951**
1.0000	0	0	**1.0000**	**1.0000**	**1.0000**	**1.0000**
0	0.0100	0	0.0650	0.0658	0.0644	**0.0662**
0	0.0625	0	0.1633	0.1699	0.1584	**0.1714**
0	0.2500	0	0.6130	**0.6224**	0.5912	0.6195
0	0.5625	0	0.9548	**0.9560**	0.9465	0.9541
0	1.0000	0	**0.9992**	**0.9992**	0.9988	0.9991
0	1.5625	0	**1.0000**	**1.0000**	**1.0000**	**1.0000**
0	0	0.0100	0.0612	0.0589	**0.0613**	0.0579
0	0	0.0625	0.1295	0.1124	**0.1301**	0.1037
0	0	0.2500	**0.4337**	0.3637	0.4332	0.3225
0	0	0.5625	0.8149	0.7409	**0.8150**	0.6865
0	0	1.0000	0.9759	0.9530	**0.9767**	0.9310
0	0	1.5625	0.9989	0.9967	**0.9991**	0.9940
0	0	2.2500	**1.0000**	0.9999	**1.0000**	0.9998
0	0	3.0625	**1.0000**	**1.0000**	**1.0000**	**1.0000**

Note. The highest value of each row is shown in bold.

Table 7: Power comparison of (i), (ii), (iii), and (iv) for the two-sample problem.

δ_1	δ_2	δ_3	(i)	(ii)	(iii)	(iv)
			$(n_1^{(\ell)}, n_2^{(\ell)}, n_3^{(\ell)}) = (20, 10, 10)$, $\ell = 1, 2$			
0.0100	0	0	0.0565	0.0576	0.0563	**0.0582**
0.0625	0	0	0.0947	0.1031	0.0927	**0.1052**
0.2500	0	0	0.2869	0.3183	0.2746	**0.3246**
0.5625	0	0	0.6332	0.6728	0.6109	**0.6787**
1.0000	0	0	0.9048	0.9228	0.8915	**0.9248**
1.5625	0	0	0.9894	0.9924	0.9867	**0.9926**
2.2500	0	0	0.9995	**0.9997**	0.9993	**0.9997**
3.0625	0	0	**1.0000**	**1.0000**	**1.0000**	**1.0000**
0	0.0100	0	0.0545	**0.0550**	0.0544	0.0549
0	0.0625	0	0.0838	0.0859	0.0831	**0.0863**
0	0.2500	0	0.2217	**0.2281**	0.2159	0.2280
0	0.5625	0	0.4938	**0.5019**	0.4805	0.4995
0	1.0000	0	0.7850	**0.7895**	0.7712	0.7856
0	1.5625	0	0.9486	**0.9497**	0.9421	0.9475
0	2.2500	0	0.9934	**0.9935**	0.9921	0.9931
0	3.0625	0	**0.9996**	**0.9996**	0.9994	0.9995
0	4.0000	0	**1.0000**	**1.0000**	**1.0000**	**1.0000**
0	0	0.0100	**0.0535**	0.0527	0.0534	0.0524
0	0	0.0625	0.0742	0.0699	**0.0745**	0.0680
0	0	0.2500	0.1647	0.1444	**0.1655**	0.1354
0	0	0.5625	0.3428	0.2962	**0.3435**	0.2738
0	0	1.0000	0.5853	0.5197	**0.5856**	0.4851
0	0	1.5625	0.8039	0.7464	**0.8043**	0.7116
0	0	2.2500	0.9343	0.9015	**0.9346**	0.8798
0	0	3.0625	0.9850	0.9733	**0.9855**	0.9645
0	0	4.0000	0.9977	0.9949	**0.9978**	0.9927
0	0	5.0625	**0.9998**	0.9994	**0.9998**	0.9990
0	0	6.2500	**1.0000**	0.9999	**1.0000**	0.9999
			$(n_1^{(\ell)}, n_2^{(\ell)}, n_3^{(\ell)}) = (30, 15, 15)$, $\ell = 1, 2$			
0.0100	0	0	0.0608	0.0620	0.0604	**0.0622**
0.0625	0	0	0.1284	0.1364	0.1258	**0.1380**
0.2500	0	0	0.4593	0.4833	0.4473	**0.4868**
0.5625	0	0	0.8596	0.8740	0.8496	**0.8751**
1.0000	0	0	0.9897	**0.9916**	0.9881	**0.9916**
1.5625	0	0	0.9998	**0.9999**	0.9998	**0.9999**
2.2500	0	0	**1.0000**	**1.0000**	**1.0000**	**1.0000**
0	0.0100	0	0.0581	0.0583	0.0580	**0.0585**
0	0.0625	0	0.1080	0.1093	0.1071	**0.1096**
0	0.2500	0	0.3491	**0.3523**	0.3433	0.3511
0	0.5625	0	0.7255	**0.7272**	0.7170	0.7241
0	1.0000	0	**0.9498**	**0.9498**	0.9464	0.9484
0	1.5625	0	**0.9968**	0.9967	0.9963	0.9965
0	2.2500	0	**0.9999**	**0.9999**	**0.9999**	**0.9999**
0	3.0625	0	**1.0000**	**1.0000**	**1.0000**	**1.0000**
0	0	0.0100	0.0554	0.0546	**0.0555**	0.0544
0	0	0.0625	0.0886	0.0836	**0.0891**	0.0819
0	0	0.2500	0.2402	0.2186	**0.2418**	0.2099
0	0	0.5625	0.5231	0.4822	**0.5245**	0.4640
0	0	1.0000	0.8075	0.7711	**0.8084**	0.7536
0	0	1.5625	0.9559	0.9407	**0.9562**	0.9326
0	0	2.2500	0.9947	0.9916	**0.9948**	0.9899
0	0	3.0625	**0.9997**	0.9994	**0.9997**	0.9992
0	0	4.0000	**1.0000**	**1.0000**	**1.0000**	**1.0000**

Note. The highest value of each row is shown in bold.

7 Concluding remarks

In this paper, we have considered the problems of testing for a mean vector and testing the equality of two mean vectors with monotone missing data. We proposed new test statistics similar to the simplified Hotelling's T^2-type test statistic for one- and two-sample problems under general-step monotone missing data, approximate upper percentiles of this new statistics by asymptotic expansion, and transformed test statistics based on Bartlett adjustment. The results of Monte Carlo simulation show that the proposed transformed test statistics $Y_{\mathrm{M}(\ell,k)}$ ($\ell = 1, 2$) were useful even for very small sample sizes. We have also presented approximate simultaneous confidence intervals for pairwise comparisons among mean vectors. We have given an approximation of the upper percentile, which is necessary when constructing approximate simultaneous confidence intervals for pairwise comparisons, and we have also evaluated the accuracy of the approximation by Monte Carlo simulation. The proposed approach was found to be useful for reasonably large sample sizes.

Acknowledgment

The first and second authors' research was partly supported by a Grant-in-Aid for Early-Career Scientists (19K20225) and a Grant-in-Aid for Scientific Research (C) (21K11795), respectively.

References

[1] Anderson, T. W. and Olkin, I. (1985). Maximum-likelihood estimation of the parameters of a multivariate normal distribution. *Linear Algebra and Its Applications*, **70**, 147–171.

[2] Jinadasa, K. G. and Tracy, D. S. (1992). Maximum likelihood estimation for multivariate normal distribution with monotone sample. *Communications in Statistics – Theory and Methods*, **21**, 41–50.

[3] Kanda, T. and Fujikoshi, Y. (1998). Some basic properties of the MLE's for a multivariate normal distribution with monotone missing data. *American Journal of Mathematical and Management Sciences*, **18**, 161–190.

[4] Kawasaki, T., Shutoh, N., and Seo, T. (2018). On the asymptotic distribution of T^2-type statistic with two-step monotone missing data. *Journal of Statistical Theory and Practice*, **12**, 657–668.

[5] Krishnamoorthy, K. and Pannala, M. K. (1999). Confidence estimation of a normal mean vector with incomplete data. *The Canadian Journal of Statistics*, **27**, 395–407.

[6] Onozawa, M., Yagi, A., and Seo, T. (2020). New test statistics for one and two mean vectors with two-step monotone missing data. *International Journal of Statistics and Probability*, **9**, 56–75.

[7] Romer, M. M. and Richards, D. St. P. (2013). Finite-sample inference with monotone incomplete multivariate normal data, III: Hotelling's T^2-statistic. *Statistical Modelling*, **13**, 431–457.

[8] Seko, N. (2012). Tests for mean vectors with two-step monotone missing data for the k-sample problem. *SUT Journal of Mathematics*, **48**, 213–229.

[9] Seko, N., Yamazaki, A., and Seo, T. (2012). Tests for mean vector with two-step monotone missing data. *SUT Journal of Mathematics*, **48**, 13–36.

[10] Yagi, A. and Seo, T. (2017). Tests for normal mean vectors with monotone incomplete data. *American Journal of Mathematical and Management Sciences*, **36**, 1–20.

[11] Yagi, A., Seo, T., and Hanusz, Z. (2018). Testing equality of two mean vectors with monotone missing data. *Technical Report No.18–07, Statistical Research Group, Hiroshima University, Hiroshima, Japan.*

[12] Yagi, A., Seo, T., and Hanusz, Z. (2019). Improved simplified T^2 test statistics for a mean vector with monotone missing data. *Communications in Statistics – Simulation and Computation*, **48**, 1606–1622.

[13] Yagi, A., Seo, T. and Hanusz, Z. (2023). Testing equality of two mean vectors with monotone incomplete data. *Communications in Statistics – Simulation and Computation*, **52**, 506–522.

[14] Yu, J., Krishnamoorthy, K., and Pannala, M. K. (2006). Two-sample inference for normal mean vectors based on monotone missing data. *Journal of Multivariate Analysis*, **97**, 2162–2176.

Appendix

A.1. Proof of Theorem 1 and Theorem 2

We consider the case of k-step monotone missing data (Theorem 2). As the distribution of $Q_i (i = 1, 2, \ldots, k)$ is given in Yagi et al. (2019), $Q_{\mathrm{M}(1,k)} = Q_1 + \sum_{i=2}^{k} R_i$ and $R_i = Q_i(1 + Q_{id})^{-1}$ $(i = 2, 3, \ldots, k)$ (from (4)), in essence, we need only focus on $(1 + Q_{id})^{-1}$. Let

$$\begin{cases} \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)(12\ldots,i-1)} = \dfrac{1}{\sqrt{N_{k-i+1}}} \boldsymbol{z}_{(12\ldots,i-1)}, \ \overline{\boldsymbol{x}}_{(12\ldots,k-i+1)i} = \dfrac{1}{\sqrt{N_{k-i+1}}} \boldsymbol{z}_i, \\ \boldsymbol{S}_{(12\ldots,k-i+1)(12\ldots i)} = \boldsymbol{I}_{p_{(12\ldots i)}} + \dfrac{1}{\sqrt{N_{k-i+1}-1}} \boldsymbol{V}_{(12\ldots i)} \end{cases}.$$

Then, assigning

$$\boldsymbol{V}_{(12\ldots i)} = \begin{pmatrix} \overbrace{\boldsymbol{V}_{11}}^{p_{(12\ldots,i-1)}} & \overbrace{\boldsymbol{V}_{12}}^{p_i} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{pmatrix} \begin{matrix} \}p_{(12\ldots,i-1)} \\ \}p_i \end{matrix},$$

and letting $\boldsymbol{z}_{(12\ldots,i-1)} = \boldsymbol{z}_1$ and $\boldsymbol{z}_i = \boldsymbol{z}_2$, $Q_i (i = 2, 3, \ldots, k)$ is expanded as follows (see Yagi et al. (2019)).

$$Q_i = \boldsymbol{z}_2'\boldsymbol{z}_2 + \frac{1}{\sqrt{N_{k-i+1}}} A_1 + \frac{1}{N_{k-i+1}} A_2 + O_p(N_{k-i+1}^{-\frac{3}{2}}),$$

where

$$\begin{aligned} A_1 =& -2\boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{z}_1 - \boldsymbol{z}_2'\boldsymbol{V}_{22}\boldsymbol{z}_2, \\ A_2 =& 2(\boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{V}_{11}\boldsymbol{z}_1 + \boldsymbol{z}_2'\boldsymbol{V}_{22}\boldsymbol{V}_{21}\boldsymbol{z}_1) + \boldsymbol{z}_1'\boldsymbol{V}_{12}\boldsymbol{V}_{21}\boldsymbol{z}_1 + \boldsymbol{z}_2'\boldsymbol{z}_2 + \boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{V}_{12}\boldsymbol{z}_2 + \boldsymbol{z}_2'\boldsymbol{V}_{22}^2\boldsymbol{z}_2. \end{aligned}$$

Given that $(1 + Q_{id})^{-1}$ is expanded as

$$(1 + Q_{id})^{-1} = 1 - \frac{1}{N_{k-i+1}} \boldsymbol{z}_1'\boldsymbol{z}_1 + O_p(N_{k-i+1}^{-\frac{3}{2}}),$$

we can expand $R_i (= Q_i(1 + Q_{id})^{-1})$ as follows.

$$R_i = \boldsymbol{z}_2'\boldsymbol{z}_2 + \frac{1}{\sqrt{N_{k-i+1}}} A_1 + \frac{1}{N_{k-i+1}} (A_2 - \boldsymbol{z}_1'\boldsymbol{z}_1\boldsymbol{z}_2'\boldsymbol{z}_2) + O_p(N_{k-i+1}^{-\frac{3}{2}}).$$

Therefore, the characteristic function of R_i can be expressed as

$$\mathrm{E}[\exp(itR_i)] = \mathrm{E}[\exp(itQ_i)] - \frac{1}{N_{k-i+1}} it\mathrm{E}\Big[\boldsymbol{z}_1'\boldsymbol{z}_1\boldsymbol{z}_2'\boldsymbol{z}_2 \exp(it\boldsymbol{z}_2'\boldsymbol{z}_2)\Big] + O(N_{k-i+1}^{-2})$$

and

$$\mathrm{E}[\exp(itQ_i)] = u^{-\frac{1}{2}p_i} + \frac{1}{N_{k-i+1}}\sum_{j=0}^{2}\beta_{j,i}u^{-\frac{1}{2}p_i-j} + O(N_{k-i+1}^{-2}), \quad i = 2, 3, \ldots, k,$$

where $u = 1 - 2it$ and

$$\beta_{0,i} = -\frac{1}{4}p_i\Big(p_i + 4p_{(12\ldots,i-1)} + 2\Big), \quad \beta_{1,i} = p_{(12\ldots,i-1)}p_i, \quad \beta_{2,i} = \frac{1}{4}p_i(p_i + 2).$$

When $i = 1$,

$$\mathrm{E}[\exp(itQ_1)] = u^{-\frac{1}{2}p_1} + \frac{1}{N_k}\sum_{j=0}^{2}\beta_{j,1}u^{-\frac{1}{2}p_1-j} + O(N_k^{-2}),$$

where

$$\beta_{0,1} = -\frac{1}{4}p_1(p_1 + 2), \quad \beta_{1,1} = 0, \quad \beta_{2,1} = -\beta_{0,1}.$$

Furthermore,

$$\mathrm{E}\Big[\boldsymbol{z}_1'\boldsymbol{z}_1\boldsymbol{z}_2'\boldsymbol{z}_2\exp(it\boldsymbol{z}_2'\boldsymbol{z}_2)\Big] = p_{(12\ldots,i-1)}p_iu^{-\frac{1}{2}p_i-1}.$$

Thus, the characteristic function of $R_i (i = 2, 3, \ldots, k)$ is

$$\mathrm{E}[\exp(itR_i)] = u^{-\frac{1}{2}p_i} + \frac{1}{N_{k-i+1}}\sum_{j=0}^{2}\gamma_{j,i}u^{-\frac{1}{2}p_i-j} + O(N_{k-i+1}^{-2}),$$

where

$$\gamma_{0,i} = -\frac{1}{4}p_i\Big(2p_{(12\ldots,i-1)} + p_i + 2\Big), \quad \gamma_{1,i} = \frac{1}{2}p_{(12\ldots,i-1)}p_i, \quad \gamma_{2,i} = \frac{1}{4}p_i(p_i + 2).$$

Because Q_1 and $R_i (i = 2, 3, \ldots, k)$ are mutually independent, the characteristic function of $Q_{\mathrm{M}(1,k)}$ can be obtained by computing

$$\mathrm{E}[\exp(itQ_{\mathrm{M}(1,k)})] = \mathrm{E}[\exp(itQ_1)]\prod_{i=2}^{k}\mathrm{E}[\exp(itR_i)].$$

Then, inverting this, we obtain the distribution function of $Q_{\mathrm{M}(1,k)}$. For Theorem 1, the proof can be obtained by setting $k = 3$. □

A.2. Proof of Theorem 3 and Theorem 4

Consider the proof of Theorem 4. For $1 \le a < b \le m$, we first consider $Q_1^{(ab)}$. Let

$$\begin{cases} \overline{\boldsymbol{x}}_{(12\ldots k)1}^{(\ell)} = \boldsymbol{\mu}_1^{(\ell)} + \dfrac{1}{\sqrt{N_k^{(\ell)}}}\boldsymbol{z}_1^{(\ell)}, & \ell = a, b \ (1 \le a < b \le m) \\ \boldsymbol{S}_{(12\ldots k)1}^{(\ell)} = \boldsymbol{I}_{p_1} + \dfrac{1}{\sqrt{N_k^{(\ell)} - 1}}\boldsymbol{V}_1^{(\ell)}, & \ell = 1, 2, \ldots, m. \end{cases}$$

Under $\boldsymbol{\mu}^{(a)} = \boldsymbol{\mu}^{(b)}$, we can expand $Q_1^{(ab)}$ as

$$Q_1^{(ab)} = \boldsymbol{z}_1'\boldsymbol{z}_1 - \frac{1}{\sqrt{N_k}}\boldsymbol{z}_1'\boldsymbol{V}_1\boldsymbol{z}_1 + \frac{1}{N_k}\Big(\boldsymbol{z}_1'\boldsymbol{V}_1^2\boldsymbol{z}_1 + \frac{m}{r}\boldsymbol{z}_1'\boldsymbol{z}_1\Big) + O_p(N_k^{-\frac{3}{2}}),$$

where

$$\boldsymbol{z}_1 = \Big(\frac{r_a r_b}{r_a + r_b}\Big)^{\frac{1}{2}}\Big(\frac{1}{\sqrt{r_a}}\boldsymbol{z}_1^{(a)} - \frac{1}{\sqrt{r_b}}\boldsymbol{z}_1^{(b)}\Big), \quad \boldsymbol{V}_1 = \sum_{\ell=1}^{m}\frac{\sqrt{r_\ell}}{r}\boldsymbol{V}_1^{(\ell)},$$
$$r_\ell = \frac{N_k^{(\ell)}}{N_k^{(1)}}, \quad \ell = 1, 2, \ldots, m, \ N_k^{(1)} = N_k, \quad r = \sum_{\ell=1}^{m} r_\ell.$$

We note that $\boldsymbol{z}_1 \sim N_{p_1}(\boldsymbol{0}, \boldsymbol{I})$. Therefore, the characteristic function of $Q_1^{(ab)}$ can be written as

$$\begin{aligned}
\mathrm{E}[\exp(itQ_1^{(ab)})] =& \mathrm{E}[\exp(it\boldsymbol{z}_1'\boldsymbol{z}_1)] + \frac{1}{\sqrt{N_k}}\mathrm{E}\Big[(-it)\boldsymbol{z}_1'\boldsymbol{V}_1\boldsymbol{z}_1\exp(it\boldsymbol{z}_1'\boldsymbol{z}_1)\Big] \\
&+ \frac{1}{N_k}\Big\{\mathrm{E}\Big[it\boldsymbol{z}_1'\boldsymbol{V}_1^2\boldsymbol{z}_1\exp(it\boldsymbol{z}_1'\boldsymbol{z}_1)\Big] + \mathrm{E}\Big[it\frac{m}{r}\boldsymbol{z}_1'\boldsymbol{z}_1\exp(it\boldsymbol{z}_1'\boldsymbol{z}_1)\Big] \\
&+ \mathrm{E}\Big[\frac{1}{2}(it)^2(\boldsymbol{z}_1'\boldsymbol{V}_1\boldsymbol{z}_1)^2\exp(it\boldsymbol{z}_1'\boldsymbol{z}_1)\Big]\Big\} + O(N_k^{-\frac{3}{2}}),
\end{aligned}$$

where $i = \sqrt{-1}$. Using the expectations described in Yagi et al. (2023, p.516), we obtain

$$\mathrm{E}[\exp(itQ_1^{(ab)})] = u^{-\frac{1}{2}p_1} + \frac{1}{N_k}\sum_{j=0}^{2}\beta_{j,1}u^{-\frac{1}{2}p_1 - j} + O(N_k^{-2}),$$

where $u = 1 - 2it$ and

$$\beta_{0,1} = -\frac{1}{4r}p_1(p_1 + 2m), \quad \beta_{1,1} = \frac{1}{2r}p_1(m-1), \quad \beta_{2,1} = \frac{1}{4r}p_1(p_1+2).$$

Next, we consider $R_i^{(ab)}$ $(i = 2, 3, \ldots, k)$. Let

$$\begin{cases}
\overline{\boldsymbol{x}}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots,i-1)} = \boldsymbol{\mu}^{(\ell)}_{(12\ldots,i-1)} + \dfrac{1}{\sqrt{N^{(\ell)}_{k-i+1}}}\boldsymbol{z}^{(\ell)}_{(12\ldots,i-1)}, \\
\overline{\boldsymbol{x}}^{(\ell)}_{(12\ldots,k-i+1)i} = \boldsymbol{\mu}^{(\ell)}_i + \dfrac{1}{\sqrt{N^{(\ell)}_{k-i+1}}}\boldsymbol{z}^{(\ell)}_i, \quad \ell = a, b \ (1 \le a < b \le m) \\
\boldsymbol{S}^{(\ell)}_{(12\ldots,k-i+1)(12\ldots i)} = \boldsymbol{I}_{p_{(12\ldots i)}} + \dfrac{1}{\sqrt{N^{(\ell)}_{k-i+1} - 1}}\boldsymbol{V}^{(\ell)}_{(12\ldots i)}, \quad \ell = 1, 2, \ldots, m.
\end{cases}$$

As with the one-sample problem, we divide $\boldsymbol{V}^{(\ell)}_{(12\ldots i)}$ as follows.

$$\boldsymbol{V}^{(\ell)}_{(12\ldots i)} = \begin{pmatrix} \overbrace{\boldsymbol{V}^{(\ell)}_{11}}^{p_{(12\ldots,i-1)}} & \overbrace{\boldsymbol{V}^{(\ell)}_{12}}^{p_i} \\ \boldsymbol{V}^{(\ell)}_{21} & \boldsymbol{V}^{(\ell)}_{22} \end{pmatrix} \begin{matrix} \}p_{(12\ldots,i-1)} \\ \}p_i \end{matrix} .$$

Then, under $\boldsymbol{\mu}^{(a)} = \boldsymbol{\mu}^{(b)}$, we can expand $Q_i^{(ab)}$ $(i = 2, 3, \ldots, k)$ as

$$Q_i^{(ab)} = \boldsymbol{z}_2'\boldsymbol{z}_2 + \frac{1}{\sqrt{N_{k-i+1}}} B_1 + \frac{1}{N_{k-i+1}} B_2 + O_p(N_{k-i+1}^{-\frac{3}{2}}),$$

where

$$\begin{aligned}
B_1 =& -2\boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{z}_1 - \boldsymbol{z}_2'\boldsymbol{V}_{22}\boldsymbol{z}_2, \\
B_2 =& \, 2(\boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{V}_{11}\boldsymbol{z}_1 + \boldsymbol{z}_2'\boldsymbol{V}_{22}\boldsymbol{V}_{21}\boldsymbol{z}_1) + \boldsymbol{z}_1'\boldsymbol{V}_{12}\boldsymbol{V}_{21}\boldsymbol{z}_1 + \boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{V}_{12}\boldsymbol{z}_2 \\
&+ \frac{m}{q}\boldsymbol{z}_2'\boldsymbol{z}_2 + \boldsymbol{z}_2'\boldsymbol{V}_{22}^2\boldsymbol{z}_2, \\
\boldsymbol{z}_1 =& \left(\frac{q_a q_b}{q_a + q_b}\right)^{\frac{1}{2}} \left(\frac{1}{\sqrt{q_a}}\boldsymbol{z}_{(12\ldots,i-1)}^{(a)} - \frac{1}{\sqrt{q_b}}\boldsymbol{z}_{(12\ldots,i-1)}^{(b)}\right), \\
\boldsymbol{z}_2 =& \left(\frac{q_a q_b}{q_a + q_b}\right)^{\frac{1}{2}} \left(\frac{1}{\sqrt{q_a}}\boldsymbol{z}_i^{(a)} - \frac{1}{\sqrt{q_b}}\boldsymbol{z}_i^{(b)}\right),
\end{aligned}$$

$$\boldsymbol{V}_{(12\ldots i)} = \sum_{\ell=1}^{m} \frac{\sqrt{q_\ell}}{q}\boldsymbol{V}_{(12\ldots i)}^{(\ell)}, \quad \boldsymbol{V}_{(12\ldots i)} = \begin{pmatrix} \overbrace{\boldsymbol{V}_{11}}^{p_{(12\ldots,i-1)}} & \overbrace{\boldsymbol{V}_{12}}^{p_i} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{pmatrix} \begin{matrix} \}p_{(12\ldots,i-1)} \\ \}p_i \end{matrix},$$

and

$$q_\ell = \frac{N_{k-i+1}^{(\ell)}}{N_{k-i+1}^{(1)}}, \quad \ell = 1, 2, \ldots, m, \ N_{k-i+1}^{(1)} = N_{k-i+1}, \quad q = \sum_{\ell=1}^{m} q_\ell.$$

We note that $\boldsymbol{z}_1 \sim N_{p_{(12\ldots,i-1)}}(\boldsymbol{0}, \boldsymbol{I})$ and $\boldsymbol{z}_2 \sim N_{p_i}(\boldsymbol{0}, \boldsymbol{I})$. The characteristic function of $Q_i^{(ab)}$ $(i = 2, 3, \ldots, k)$ can be written as

$$\begin{aligned}
&\mathrm{E}[\exp(itQ_i^{(ab)})] \\
&= \mathrm{E}[\exp(it\boldsymbol{z}_2'\boldsymbol{z}_2)] + \frac{1}{\sqrt{N_{k-i+1}}}\mathrm{E}\Big[(-it)(2\boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{z}_1 + \boldsymbol{z}_2'\boldsymbol{V}_{22}\boldsymbol{z}_2)\exp(it\boldsymbol{z}_2'\boldsymbol{z}_2)\Big] \\
&\quad + \frac{1}{N_{k-i+1}}\mathrm{E}\Big[it\Big\{2(\boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{V}_{11}\boldsymbol{z}_1 + \boldsymbol{z}_2'\boldsymbol{V}_{22}\boldsymbol{V}_{21}\boldsymbol{z}_1) + \boldsymbol{z}_1'\boldsymbol{V}_{12}\boldsymbol{V}_{21}\boldsymbol{z}_1 \\
&\qquad + \boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{V}_{12}\boldsymbol{z}_2 + \frac{m}{q}\boldsymbol{z}_2'\boldsymbol{z}_2 + \boldsymbol{z}_2'\boldsymbol{V}_{22}^2\boldsymbol{z}_2\Big\}\exp(it\boldsymbol{z}_2'\boldsymbol{z}_2) \\
&\qquad + \frac{1}{2}(it)^2\Big\{4(\boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{z}_1)^2 + 4(\boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{z}_1)(\boldsymbol{z}_2'\boldsymbol{V}_{22}\boldsymbol{z}_2) + (\boldsymbol{z}_2'\boldsymbol{V}_{22}\boldsymbol{z}_2)^2\Big\}\exp(it\boldsymbol{z}_2'\boldsymbol{z}_2)\Big] \\
&\quad + O(N_{k-i+1}^{-\frac{3}{2}}).
\end{aligned}$$

After some calculations using the expectations described in Yagi et al. (2023, p.518), we obtain the characteristic function of $Q_i^{(ab)}$ as

$$\mathrm{E}[\exp(itQ_i^{(ab)})] = u^{-\frac{1}{2}p_i} + \frac{1}{N_{k-i+1}}\sum_{j=0}^{2}\beta_{j,i}u^{-\frac{1}{2}p_i - j} + O(N_{k-i+1}^{-2}), \quad i = 2, 3, \ldots, k,$$

where

$$\beta_{0,i} = -\frac{1}{4q}p_i(4p_{(12\ldots,i-1)} + p_i + 2m), \quad \beta_{1,i} = \frac{1}{2q}p_i(2p_{(12\ldots,i-1)} + m - 1),$$
$$\beta_{2,i} = \frac{1}{4q}p_i(p_i + 2).$$

Further, $R_i^{(ab)} (= Q_i^{(ab)}(1 + Q_{id}^{(ab)})^{-1})$ can be expanded as

$$R_i^{(ab)} = \boldsymbol{z}_2'\boldsymbol{z}_2 + \frac{1}{\sqrt{N_{k-i+1}}}B_1 + \frac{1}{N_{k-i+1}}(B_2 - \frac{1}{q}\boldsymbol{z}_1'\boldsymbol{z}_1\boldsymbol{z}_2'\boldsymbol{z}_2) + O_p(N_{k-i+1}^{-\frac{3}{2}}).$$

Therefore, the characteristic function of $R_i^{(ab)}$ can be expressed as

$$\mathrm{E}[\exp(itR_i^{(ab)})] = \mathrm{E}[\exp(itQ_i^{(ab)})] - \frac{1}{N_{k-i+1}}it\frac{1}{q}\mathrm{E}\Big[\boldsymbol{z}_1'\boldsymbol{z}_1\boldsymbol{z}_2'\boldsymbol{z}_2\exp(it\boldsymbol{z}_2'\boldsymbol{z}_2)\Big] + O(N_{k-i+1}^{-2}).$$

Thus, as with the one-sample problem case, the characteristic function of $R_i^{(ab)}$ is

$$\mathrm{E}[\exp(itR_i^{(ab)})] = u^{-\frac{1}{2}p_i} + \frac{1}{N_{k-i+1}}\sum_{j=0}^{2}\gamma_{j,i}u^{-\frac{1}{2}p_i - j} + O(N_{k-i+1}^{-2}),$$

where

$$\gamma_{0,i} = -\frac{1}{4q}p_i\Big(2p_{(12\ldots,i-1)} + p_i + 2m\Big), \ \gamma_{1,i} = \frac{1}{2q}p_{(12\ldots,i-1)}(p_i + m - 1), \ \gamma_{2,i} = \frac{1}{4q}p_i(p_i + 2).$$

Because $\mathrm{E}[\exp(itQ_{\mathrm{M}(1,k)}^{(ab)})] = \mathrm{E}[\exp(itQ_1^{(ab)})]\prod_{i=2}^{k}\mathrm{E}[\exp(itR_i^{(ab)})]$, we can obtain the characteristic function and distribution function of $Q_{\mathrm{M}(1,k)}^{(ab)}$. For Theorem 3, the proof can be obtained by setting $m = 2$. □