Tests for Multivariate Kurtosis with Two- and Three-Step Monotone Missing Data

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Abstract

In this paper, we consider the test statistic for multivariate kurtosis and its percentiles of the null distribution to test for multivariate normality with monotone missing data. In particular, we formulate a test statistic for which the normal approximation in the case of two-step monotone missing data is given by the expectation and variance approximated by linear approximation. Furthermore, we extend this statistic to the case of three-step monotone missing data. Specifically, we define multivariate sample kurtosis for three-step monotone missing data, and formulate a new test statistic that uses information approximated by linear interpolation. Finally, we investigate the accuracy and behavior of the normal approximation by a Monte Carlo simulation.

Key Words and Phrases. Fourth-order moment, linear interpolation, multivariate normality test, multivariate sample kurtosis, normal approximation.

1 Introduction

Most inferential procedures in multivariate statistical data analyses are based upon multivariate normal distributions. Assuming that the observation vectors are independent and have a multivariate normal distribution, the theoretical properties of the statistical procedures developed under multivariate normality can be considered effective. Therefore, the multivariate normality (MVN) test problem represents a crucial analytical task. Many studies have previously been concluded on the MVN test problem: see, for example, Mardia (1970, 1974), Malkovich and Afifi (1973), Srivastava (1984), Koziol (1989), Henze and Zirkler (1990). Recently, Enomoto et al. (2020) gave a normalizing transformation statistic for Mardia's multivariate sample kurtosis test statistic. Furthermore the problem of missing data is often encountered for various reasons in many practical situations, especially in epidemiology and biostatistics, leading to the MVN test problem with missing data. In this study, we specifically examined the with multivariate kurtosis test in the case of monotone missing data. Maximum likelihood estimators (MLEs) for the mean vector and the covariance matrix in the case of monotone missing data patterns have previously been obtained as closed-form expressions by Jinadasa and Tracy (1992) and Kanda and Fujikoshi (1998). Chang and Richards (2009) and Seko et al. (2012) defined Hotelling's T^2 type statistic in the case of a two-step monotone missing data pattern. Seko et al. (2012) approximated the upper percentiles of this statistic using linear interpolation based on the complete data parts. For three-step monotone missing data, refer to the study published by Yagi and Seo (2015). In the present study, we define sample measures of multivariate kurtosis given two- and three-step monotone patterns of missing observations in the data. Furthermore, we formulate a test statistic by applying the linear interpolation approximation to the expectation and variance of the multivariate sample kurtosis. For two-step monotone missing data, Kurita and Seo (2022) discussed the multivariate sample kurtosis and approximated its expectation and variance using asymptotic expansion. Yamada et al. (2015) gave another definition of multivariate sample kurtosis along with a corresponding test statistic for the case of two-step monotone missing data. The rest of this paper is organized as follows. Section 2 introduces Mardia's multivariate sample kurtosis and the test statistic for the complete data case. In Sections 3 and 4, we define measures of multivariate sample kurtosis with two- and three-step monotone missing data, respectively. Furthermore, we present new test statistics using the linear interpolation approximation. In Section 5, we investigate the accuracy and approximate behavior of the normal approximation through a Monte Carlo simulation. Finally, we conclude the paper in Section 6.

2 Mardia's multivariate kurtosis

Consider a p-variate random vector $\boldsymbol{x}$ from a multivariate distribution with mean vector $\boldsymbol{\mu}$ and positive definite covariance matrix $\boldsymbol{\Sigma}$. Mardia (1970) defined the population measure of multivariate kurtosis as

$$\beta_{2,p} = \mathrm{E}[\{(\boldsymbol{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\}^2].$$

When $p = 1$, $\beta_{2,p}$ reduces to the ordinary univariate measure, which is invariant under affine transformation. For the multivariate normal distribution, it is well-known that $\beta_{2,p} = p(p+2)$. Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_N$ be random sample vectors from a p-variate population with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Then, the sample measure of multivariate kurtosis is defined as

$$b_{2,p} = \frac{1}{N} \sum_{i=1}^{N} \{(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top \boldsymbol{S}^{-1} (\boldsymbol{x}_i - \overline{\boldsymbol{x}})\}^2,$$

where $\overline{\boldsymbol{x}} = (1/N) \sum_{i=1}^{N} \boldsymbol{x}_i$ and $\boldsymbol{S} = (1/N) \sum_{i=1}^{N} (\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top$.

Mardia (1974) obtained the following exact mean and variance of $b_{2,p}$ for a population $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

$$\begin{aligned}
\mathrm{E}[b_{2,p}] &= p(p+2)\frac{N-1}{N+1}, \\
\mathrm{Var}[b_{2,p}] &= 8p(p+2)\frac{(N-3)(N-p-1)(N-p+1)}{(N+1)^2(N+3)(N+5)}.
\end{aligned}$$

Consequently, the following test statistics were given to test the MVN:

$$\begin{aligned}
Z_{\mathrm{M}} &= \frac{N^{\frac{1}{2}}\{b_{2,p} - p(p+2)\}}{\left\{\dfrac{8}{N}p(p+2)\right\}^{\frac{1}{2}}}, \\
Z_{\mathrm{M}}^* &= \frac{\{(N+1)b_{2,p} - p(p+2)(N-1)\}\{(N+3)(N+5)\}^{\frac{1}{2}}}{\{8p(p+2)(N-3)(N-p-1)(N-p+1)\}^{\frac{1}{2}}},
\end{aligned}$$

where Z_{M} and Z_{M}^* are asymptotically distributed as $N(0,1)$. See, Mardia (1970, 1974).

3 Two-step monotone missing data

3.1 Multivariate sample kurtosis $b_{2,p}^{(2)}$

Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{N_1}$ be N_1 p-variate random sample vectors and $\boldsymbol{x}_{1,N_1+1}, \dots, \boldsymbol{x}_{1,N}$ be N_2 p_1-variate random sample vectors. Two-step monotone missing data can then be expressed as follows:

$$\begin{pmatrix}
\boldsymbol{x}_{1,1}^\top & \boldsymbol{x}_{2,1}^\top \\
\vdots & \vdots \\
\boldsymbol{x}_{1,N_1}^\top & \boldsymbol{x}_{2,N_1}^\top \\
\boldsymbol{x}_{1,N_1+1}^\top & * \\
\vdots & \vdots \\
\boldsymbol{x}_{1,N}^\top & *
\end{pmatrix},$$

where $\boldsymbol{x}_i = (\boldsymbol{x}_{1,i}^\top, \boldsymbol{x}_{2,i}^\top)^\top$, $i = 1, \ldots, N_1$, $N = N_1 + N_2$, $p = p_1 + p_2$, and "$*$" indicates a missing p_2-dimensional vector. We assume that the data are missing completely at random (MCAR). Furthermore, we assume a multivariate normal distribution for the two-step monotone missing data, i.e.,

$$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N_1} \overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \quad \boldsymbol{x}_{1,N_1+1}, \ldots, \boldsymbol{x}_{1,N} \overset{i.i.d.}{\sim} N_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}),$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

The following notation is used for preparation: $\boldsymbol{\mu}_{2\cdot1} = \boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1$ and $\boldsymbol{\Sigma}_{22\cdot1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$. The sample measure of multivariate kurtosis is defined by Kurita and Seo (2022) as follows:

$$b_{2,p}^{(2)} = R_1^{(2)} + R_2^{(2)} + R_{12}^{(2)},$$

where

$$R_1^{(2)} = \frac{1}{N}\sum_{i=1}^{N} U_{1,i}^2, \; R_2^{(2)} = \frac{1}{N_1}\sum_{i=1}^{N_1} U_{2\cdot1,i}^2, \; R_{12}^{(2)} = \frac{2}{N_1}\sum_{i=1}^{N_1} U_{1,i}U_{2\cdot1,i},$$

and

$$U_{1,i} = (\boldsymbol{x}_{1,i} - \widehat{\boldsymbol{\mu}}_1)^\top \widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\boldsymbol{x}_{1,i} - \widehat{\boldsymbol{\mu}}_1), \; U_{2\cdot1,i} = (\boldsymbol{x}_{2\cdot1,i} - \widehat{\boldsymbol{\mu}}_{2\cdot1})^\top \widehat{\boldsymbol{\Sigma}}_{22\cdot1}^{-1}(\boldsymbol{x}_{2\cdot1,i} - \widehat{\boldsymbol{\mu}}_{2\cdot1}),$$

$\boldsymbol{x}_{2\cdot1,i} = \boldsymbol{x}_{2,i} - \widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\boldsymbol{x}_{1,i}$, $\widehat{\boldsymbol{\mu}}_1$, $\widehat{\boldsymbol{\Sigma}}_{11}$, $\widehat{\boldsymbol{\mu}}_{2\cdot1}$, and $\widehat{\boldsymbol{\Sigma}}_{22\cdot1}$ are MLEs of $\boldsymbol{\mu}_1$, $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\mu}_{2\cdot1}$, and $\boldsymbol{\Sigma}_{22\cdot1}$, respectively. The MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are given by Kanda and Fujikoshi (1998). We then use the following notation for some definitions:

$$\overline{\boldsymbol{x}}_{(1)} = \frac{1}{N_1}\sum_{i=1}^{N_1}\boldsymbol{x}_i = \begin{pmatrix} \overline{\boldsymbol{x}}_{(1),1} \\ \overline{\boldsymbol{x}}_{(1),2} \end{pmatrix}, \quad \boldsymbol{S}_{(1)} = \frac{1}{N_1}\sum_{i=1}^{N_1}(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{(1)})(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{(1)})^\top = \begin{pmatrix} \boldsymbol{S}_{(1),11} & \boldsymbol{S}_{(1),12} \\ \boldsymbol{S}_{(1),21} & \boldsymbol{S}_{(1),22} \end{pmatrix},$$

$$\overline{\boldsymbol{x}}_{(2)} = \frac{1}{N_2}\sum_{i=N_1+1}^{N}\boldsymbol{x}_{1,i}, \quad \boldsymbol{S}_{(2)} = \frac{1}{N_2}\sum_{i=N_1+1}^{N}(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_{(2)})(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_{(2)})^\top,$$

$$\overline{\boldsymbol{x}}_T = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_{1,i}, \quad \boldsymbol{S}_T = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T)(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T)^\top.$$

Then we note that

$$\widehat{\boldsymbol{\mu}}_1 = \overline{\boldsymbol{x}}_T, \quad \widehat{\boldsymbol{\Sigma}}_{11} = \boldsymbol{S}_T, \; \boldsymbol{x}_{2\cdot1,i} = \boldsymbol{x}_{2,i} - \boldsymbol{S}_{(1),21}\boldsymbol{S}_{(1),11}^{-1}\boldsymbol{x}_{1,i},$$

$$\widehat{\boldsymbol{\mu}}_{2\cdot1} = \overline{\boldsymbol{x}}_{(1),2} - \boldsymbol{S}_{(1),21}\boldsymbol{S}_{(1),11}^{-1}\overline{\boldsymbol{x}}_{(1),1}, \; \widehat{\boldsymbol{\Sigma}}_{22\cdot1} = \boldsymbol{S}_{(1),22} - \boldsymbol{S}_{(1),21}\boldsymbol{S}_{(1),11}^{-1}\boldsymbol{S}_{(1),12}.$$

Furthermore, $\overline{\boldsymbol{x}}_T$ and $\boldsymbol{S}_T$ can be written as

$$\overline{\boldsymbol{x}}_T = \tau_1 \overline{\boldsymbol{x}}_{(1),1} + (1-\tau_1)\overline{\boldsymbol{x}}_{(2)}, \quad \boldsymbol{S}_T = \tau_1 \boldsymbol{S}_{(1),11} + (1-\tau_1)\boldsymbol{S}_{(2)} + \tau_1(1-\tau_1)(\overline{\boldsymbol{x}}_{(1),1} - \overline{\boldsymbol{x}}_{(2)})(\overline{\boldsymbol{x}}_{(1),1} - \overline{\boldsymbol{x}}_{(2)})^\top,$$

respectively, where τ_1 is a constant such that $N_1 = \tau_1 N,\ 0 < \tau_1 < 1$. By substituting these MLEs into $U_{1,i}$ and $U_{2\cdot 1,i}$, we can write

$$\begin{aligned} U_{1,i} &= (\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T)^\top \boldsymbol{S}_T^{-1}(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T), \\ U_{2\cdot 1,i} &= \{\boldsymbol{x}_{2,i} - \overline{\boldsymbol{x}}_{(1),2} - \boldsymbol{S}_{(1),21}\boldsymbol{S}_{(1),11}^{-1}(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_{(1),1})\}^\top (\boldsymbol{S}_{(1),22} - \boldsymbol{S}_{(1),21}\boldsymbol{S}_{(1),11}^{-1}\boldsymbol{S}_{(1),12})^{-1} \\ &\quad \times \{\boldsymbol{x}_{2,i} - \overline{\boldsymbol{x}}_{(1),2} - \boldsymbol{S}_{(1),21}\boldsymbol{S}_{(1),11}^{-1}(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_{(1),1})\} \end{aligned}$$

respectively.

3.2 Kurtosis test statistic

Because the two-step monotone dataset is between the complete data $(N_1 \times p)$ and the complete data $(N \times p)$, the linear interpolation can be approximated, assuming that the expectation and variance for the two-step monotone missing data fall within the same range, where we assume that missing elements of the complete data $(N \times p)$ are filled in (See Figure 1). Then, the approximate expectation and variance of $b_{2,p}^{(2)}$ are given by

$$\begin{aligned} m_{\mathrm{L}}^{(2)} &= p_1(p+2)\frac{N-1}{N+1} + p_2(p+2)\frac{\tau_1 N - 1}{\tau_1 N + 1}, \\ (\nu_{\mathrm{L}}^{(2)})^2 &= 8p_1(p+2)\frac{(N-3)(N-p-1)(N-p+1)}{(N+1)^2(N+3)(N+5)} + 8p_2(p+2)\frac{(\tau_1 N-3)(\tau_1 N-p-1)(\tau_1 N-p+1)}{(\tau_1+1)^2(\tau_1 N+3)(\tau_1 N+5)}. \end{aligned}$$

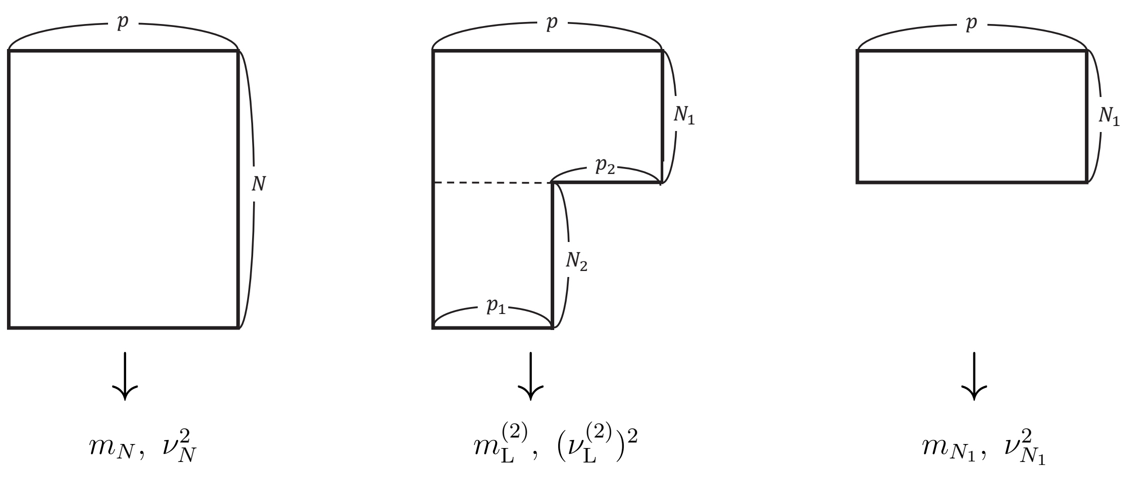

Figure 1. Approximate expectation and variance of multivariate sample kurtosis $b_{2,p}^{(2)}$

We note from Figure 1 that $m_{\mathrm{L}}^{(2)}$ and $(\nu_{\mathrm{L}}^{(2)})^2$ may fall between $m_N,\ \nu_N^2$ and $m_{N_1},\ \nu_{N_1}^2$, respectively, where

$$
\begin{aligned}
m_N &= p(p+2)\frac{N-1}{N+1},\ \nu_N^2 = 8p(p+2)\frac{(N-3)(N-p-1)(N-p+1)}{(N+1)^2(N+3)(N+5)},\\
m_{N_1} &= p(p+2)\frac{N_1-1}{N_1+1},\ \nu_{N_1}^2 = 8p(p+2)\frac{(N_1-3)(N_1-p-1)(N_1-p+1)}{(N_1+1)^2(N_1+3)(N_1+5)}.
\end{aligned}
$$

Therefore, we propose a new test statistic given by

$$
Z_{\mathrm{MML}}^{(2)} = \frac{b_{2,p}^{(2)} - m_{\mathrm{L}}^{(2)}}{\nu_{\mathrm{L}}^{(2)}},
$$

where $Z_{\mathrm{MML}}^{(2)}$ is asymptotically distributed as $N(0,1)$.

4 Three-step monotone missing data

4.1 Multivariate sample kurtosis $b_{2,p}^{(3)}$

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N_1}$ be N_1 p-variate random sample vectors, $\boldsymbol{x}_{(12),N_1+1}, \ldots, \boldsymbol{x}_{(12),N_1+N_2}$ be N_2 (p_1+p_2)-variate random sample vectors, and $\boldsymbol{x}_{1,N_1+N_2+1}, \ldots, \boldsymbol{x}_{1,N}$ be N_3 p_1-variate random sample vectors. Such a dataset has three-step monotone missing data:

$$
\left(\begin{array}{ccc}
\boldsymbol{x}_{1,1}^\top & \boldsymbol{x}_{2,1}^\top & \boldsymbol{x}_{3,1}^\top \\
\vdots & \vdots & \vdots \\
\boldsymbol{x}_{1,N_1}^\top & \boldsymbol{x}_{2,N_1}^\top & \boldsymbol{x}_{3,N_1}^\top \\
\boldsymbol{x}_{1,N_1+1}^\top & \boldsymbol{x}_{2,N_1+1}^\top & * \\
\vdots & \vdots & \vdots \\
\boldsymbol{x}_{1,N_1+N_2}^\top & \boldsymbol{x}_{2,N_1+N_2}^\top & \vdots \\
\boldsymbol{x}_{1,N_1+N_2+1}^\top & * & \vdots \\
\vdots & \vdots & \vdots \\
\boldsymbol{x}_{1,N}^\top & * & *
\end{array}\right),
$$

where $N = N_1 + N_2 + N_3$, $p = p_1 + p_2 + p_3$, and " $*$ " indicates a missing vector. Let $\boldsymbol{x}_i = (\boldsymbol{x}_{1,i}^\top, \boldsymbol{x}_{2,i}^\top, \boldsymbol{x}_{3,i}^\top)^\top,\ i = 1, \ldots, N_1$ be a random vector from a p-variate distribution with expectation $\boldsymbol{\mu}$ and positive definite covariance matrix $\boldsymbol{\Sigma}$, and let $\boldsymbol{x}_{(12),i} = (\boldsymbol{x}_{1,i}^\top, \boldsymbol{x}_{2,i}^\top)^\top,\ i = N_1+1, \ldots, N_1+N_2$ be a random vector from a (p_1+p_2)-variate distribution with expectation $\boldsymbol{\mu}_{(12)}$ and positive definite covariance matrix $\boldsymbol{\Sigma}_{(12)(12)}$. Furthermore, let $\boldsymbol{x}_{1,i},\ i = N_1+N_2+1, \ldots, N$ be a random vector from a p_1-variate distribution with expectation $\boldsymbol{\mu}_1$ and positive definite covariance matrix $\boldsymbol{\Sigma}_{11}$, where the decompositions of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)} \\ \boldsymbol{\mu}_3 \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \left(\begin{array}{cc|c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \boldsymbol{\Sigma}_{13} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \boldsymbol{\Sigma}_{23} \\ \hline \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{\Sigma}_{(12)(12)} & \boldsymbol{\Sigma}_{(12)3} \\ \hline \boldsymbol{\Sigma}_{3(12)} & \boldsymbol{\Sigma}_{33} \end{array}\right),$$

respectively. In addition to the notation in Subsection 3.1, we define $\boldsymbol{\mu}_{3\cdot 12} = \boldsymbol{\mu}_3 - \boldsymbol{\Sigma}_{3(12)}\boldsymbol{\Sigma}_{(12)(12)}^{-1}\boldsymbol{\mu}_{(12)}$, $\boldsymbol{\Sigma}_{33\cdot 12} = \boldsymbol{\Sigma}_{33} - \boldsymbol{\Sigma}_{3(12)}\boldsymbol{\Sigma}_{(12)(12)}^{-1}\boldsymbol{\Sigma}_{(12)3}$. Then, the sample measure of multivariate kurtosis in the case of three-step monotone missing data can be defined as

$$b_{2,p}^{(3)} = \sum_{j=1}^{3} R_j^{(3)} + \sum_{\substack{j=1 \\ j<k}}^{3}\sum_{k=1}^{3} R_{jk}^{(3)},$$

where

$$R_1^{(3)} = \frac{1}{N}\sum_{i=1}^{N} U_{1,i}^2,\ R_2^{(3)} = \frac{1}{N_1+N_2}\sum_{i=1}^{N_1+N_2} U_{2\cdot 1,i}^2,\ R_3^{(3)} = \frac{1}{N_1}\sum_{i=1}^{N_1} U_{3\cdot 12,i}^2,$$
$$R_{12}^{(3)} = \frac{2}{N_1+N_2}\sum_{i=1}^{N_1+N_2} U_{1,i}U_{2\cdot 1,i},\ R_{13}^{(3)} = \frac{2}{N_1}\sum_{i=1}^{N_1} U_{1,i}U_{3\cdot 12,i},\ R_{23}^{(3)} = \frac{2}{N_1}\sum_{i=1}^{N_1} U_{2\cdot 1,i}U_{3\cdot 12,i},$$

and

$$U_{3\cdot 12,i} = (\boldsymbol{x}_{3\cdot 12,i} - \widehat{\boldsymbol{\mu}}_{3\cdot 12})^\top \widehat{\boldsymbol{\Sigma}}_{33\cdot 12}^{-1}(\boldsymbol{x}_{3\cdot 12,i} - \widehat{\boldsymbol{\mu}}_{3\cdot 12}),\ \boldsymbol{x}_{3\cdot 12,i} = \boldsymbol{x}_{3,i} - \widehat{\boldsymbol{\Sigma}}_{3(12)}\widehat{\boldsymbol{\Sigma}}_{(12)(12)}^{-1}\boldsymbol{x}_{(12),i}.$$

We note that $\widehat{\boldsymbol{\mu}}_{3\cdot 12}$ and $\widehat{\boldsymbol{\Sigma}}_{33\cdot 12}$ are MLEs of $\boldsymbol{\mu}_{3\cdot 12}$, and $\boldsymbol{\Sigma}_{33\cdot 12}$, respectively. We also note that the definitions of $U_{1,i}$ and $U_{2\cdot 1,i}$ are equivalent to those for the two-step case in Section 3. The MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ for the three-step case were defined by Kanda and Fujikoshi (1998). First, we define the following notation:

$$\overline{\boldsymbol{x}}_{(1)} = \frac{1}{N_1}\sum_{i=1}^{N_1}\boldsymbol{x}_i = \begin{pmatrix}\overline{\boldsymbol{x}}_{(1),1}\\ \overline{\boldsymbol{x}}_{(1),2}\\ \overline{\boldsymbol{x}}_{(1),3}\end{pmatrix} = \begin{pmatrix}\overline{\boldsymbol{x}}_{(1),(12)}\\ \overline{\boldsymbol{x}}_{(1),3}\end{pmatrix},\ \overline{\boldsymbol{x}}_{(2)} = \frac{1}{N_2}\sum_{i=N_1+1}^{N_1+N_2}\boldsymbol{x}_{(12),i} = \begin{pmatrix}\overline{\boldsymbol{x}}_{(2),1}\\ \overline{\boldsymbol{x}}_{(2),2}\end{pmatrix},$$

$$\overline{\boldsymbol{x}}_{(3)} = \frac{1}{N_3}\sum_{i=N_1+N_2+1}^{N}\boldsymbol{x}_{1,i},$$

$$\boldsymbol{S}_{(1)} = \frac{1}{N_1}\sum_{i=1}^{N_1}(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{(1)})(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{(1)})^\top$$

$$= \left(\begin{array}{cc|c}\boldsymbol{S}_{(1),11} & \boldsymbol{S}_{(1),12} & \boldsymbol{S}_{(1),13}\\ \boldsymbol{S}_{(1),21} & \boldsymbol{S}_{(1),22} & \boldsymbol{S}_{(1),23}\\ \hline \boldsymbol{S}_{(1),31} & \boldsymbol{S}_{(1),32} & \boldsymbol{S}_{(1),33}\end{array}\right) = \left(\begin{array}{c|c}\boldsymbol{S}_{(1),(12)(12)} & \boldsymbol{S}_{(1),(12)3}\\ \hline \boldsymbol{S}_{(1),3(12)} & \boldsymbol{S}_{(1),33}\end{array}\right),$$

$$\boldsymbol{S}_{(2)} = \frac{1}{N_2}\sum_{i=N_1+1}^{N_1+N_2}(\boldsymbol{x}_{(12),i} - \overline{\boldsymbol{x}}_{(2)})(\boldsymbol{x}_{(12),i} - \overline{\boldsymbol{x}}_{(2)})^\top = \begin{pmatrix}\boldsymbol{S}_{(2),11} & \boldsymbol{S}_{(2),12}\\ \boldsymbol{S}_{(2),21} & \boldsymbol{S}_{(2),22}\end{pmatrix},$$

$$\boldsymbol{S}_{(3)} = \frac{1}{N_3}\sum_{i=N_1+N_2+1}^{N}(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_{(3)})(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_{(3)})^\top,$$

$$\overline{\boldsymbol{x}}_{T(12)} = \frac{1}{N_1+N_2}\sum_{i=1}^{N_1+N_2}\boldsymbol{x}_{(12),i} = \begin{pmatrix}\overline{\boldsymbol{x}}_{T(12),1}\\ \overline{\boldsymbol{x}}_{T(12),2}\end{pmatrix},\ \overline{\boldsymbol{x}}_T = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_{1,i},$$

$$\boldsymbol{S}_{T(12)} = \frac{1}{N_1+N_2}\sum_{i=1}^{N_1+N_2}(\boldsymbol{x}_{(12),i} - \overline{\boldsymbol{x}}_{T(12)})(\boldsymbol{x}_{(12),i} - \overline{\boldsymbol{x}}_{T(12)})^\top = \begin{pmatrix}\boldsymbol{S}_{T(12),11} & \boldsymbol{S}_{T(12),12}\\ \boldsymbol{S}_{T(12),21} & \boldsymbol{S}_{T(12),22}\end{pmatrix},$$

$$\boldsymbol{S}_T = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T)(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T)^\top.$$

Then, we have

$$\widehat{\boldsymbol{\mu}}_1 = \overline{\boldsymbol{x}}_T,\quad \widehat{\boldsymbol{\Sigma}}_{11} = \boldsymbol{S}_T,$$

$$\widehat{\boldsymbol{\mu}}_{2\cdot 1} = \overline{\boldsymbol{x}}_{T(12),2} - \boldsymbol{S}_{T(12),21}\boldsymbol{S}_{T(12),11}^{-1}\overline{\boldsymbol{x}}_{T(12),1},\quad \widehat{\boldsymbol{\Sigma}}_{22\cdot 1} = \boldsymbol{S}_{T(12),22} - \boldsymbol{S}_{T(12),21}\boldsymbol{S}_{T(12),11}^{-1}\boldsymbol{S}_{T(12),12},$$

$$\widehat{\boldsymbol{\mu}}_{3\cdot 12} = \overline{\boldsymbol{x}}_{(1),3} - \boldsymbol{S}_{(1),3(12)}\boldsymbol{S}_{(1),(12)(12)}^{-1}\overline{\boldsymbol{x}}_{(1),(12)},\quad \widehat{\boldsymbol{\Sigma}}_{33\cdot 12} = \boldsymbol{S}_{(1),33} - \boldsymbol{S}_{(1),3(12)}\boldsymbol{S}_{(1),(12)(12)}^{-1}\boldsymbol{S}_{(1),(12)3}.$$

Furthermore, we note that

$$\overline{\boldsymbol{x}}_T = \tau_1\overline{\boldsymbol{x}}_{(1),1} + \tau_2\overline{\boldsymbol{x}}_{(2),1} + \tau_3\overline{\boldsymbol{x}}_{(3)},\ \overline{\boldsymbol{x}}_{T(12)} = \frac{\tau_1}{\tau_1+\tau_2}\overline{\boldsymbol{x}}_{(1),(12)} + \frac{\tau_2}{\tau_1+\tau_2}\overline{\boldsymbol{x}}_{(2)},$$

$$\boldsymbol{S}_T = \tau_1\boldsymbol{S}_{(1),11} + \tau_2\boldsymbol{S}_{(2),11} + \tau_3\boldsymbol{S}_{(3)} + \tau_1\tau_2(\overline{\boldsymbol{x}}_{(1),1} - \overline{\boldsymbol{x}}_{(2),1})(\overline{\boldsymbol{x}}_{(1),1} - \overline{\boldsymbol{x}}_{(2),1})^\top$$
$$+ \tau_1\tau_3(\overline{\boldsymbol{x}}_{(1),1} - \overline{\boldsymbol{x}}_{(3)})(\overline{\boldsymbol{x}}_{(1),1} - \overline{\boldsymbol{x}}_{(3)})^\top + \tau_2\tau_3(\overline{\boldsymbol{x}}_{(2),1} - \overline{\boldsymbol{x}}_{(3)})(\overline{\boldsymbol{x}}_{(2),1} - \overline{\boldsymbol{x}}_{(3)})^\top,$$

$$\boldsymbol{S}_{T(12)} = \frac{\tau_1}{\tau_1+\tau_2}\boldsymbol{S}_{(1),(12)(12)} + \frac{\tau_2}{\tau_1+\tau_2}\boldsymbol{S}_{(2)} + \frac{\tau_1\tau_2}{(\tau_1+\tau_2)^2}(\overline{\boldsymbol{x}}_{(1),(12)} - \overline{\boldsymbol{x}}_{(2)})(\overline{\boldsymbol{x}}_{(1),(12)} - \overline{\boldsymbol{x}}_{(2)})^\top,$$

where τ_1, τ_2, and τ_3 are constants such that $N_1 = \tau_1 N,\ N_2 = \tau_2 N,\ N_3 = \tau_3 N,\ 0 < \tau_1 < 1,\ 0 < \tau_2 < 1$, and $0 < \tau_3 < 1$. By substituting these MLEs into $U_{1,i}$, $U_{2\cdot 1,i}$, and $U_{3\cdot 12,i}$, we can write

$$U_{1,i} = (\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T)^\top \boldsymbol{S}_T^{-1} (\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T),$$
$$U_{2\cdot 1,i} = (\boldsymbol{x}_{2\cdot 1,i} - \widehat{\boldsymbol{\mu}}_{2\cdot 1})^\top \widehat{\boldsymbol{\Sigma}}_{22\cdot 1}^{-1} (\boldsymbol{x}_{2\cdot 1,i} - \widehat{\boldsymbol{\mu}}_{2\cdot 1}),$$
$$U_{3\cdot 12,i} = (\boldsymbol{x}_{3\cdot 12,i} - \widehat{\boldsymbol{\mu}}_{3\cdot 12})^\top \widehat{\boldsymbol{\Sigma}}_{33\cdot 12}^{-1} (\boldsymbol{x}_{3\cdot 12,i} - \widehat{\boldsymbol{\mu}}_{3\cdot 12}),$$

where

$$\boldsymbol{x}_{2\cdot 1,i} = \boldsymbol{x}_{2,i} - \boldsymbol{S}_{T(12),21} \boldsymbol{S}_{T(12),11}^{-1} \boldsymbol{x}_{1,i}, \quad \widehat{\boldsymbol{\mu}}_{2\cdot 1} = \overline{\boldsymbol{x}}_{T(12),2} - \boldsymbol{S}_{T(12),21} \boldsymbol{S}_{T(12),11}^{-1} \overline{\boldsymbol{x}}_{T(12),1},$$
$$\boldsymbol{x}_{3\cdot 12,i} = \boldsymbol{x}_{3,i} - \boldsymbol{S}_{(1),3(12)} \boldsymbol{S}_{(1),(12)(12)}^{-1} \boldsymbol{x}_{(12),i}, \quad \widehat{\boldsymbol{\mu}}_{3\cdot 12} = \overline{\boldsymbol{x}}_{(1),3} - \boldsymbol{S}_{(1),3(12)} \boldsymbol{S}_{(1),(12)(12)}^{-1} \overline{\boldsymbol{x}}_{(1),(12)},$$
$$\widehat{\boldsymbol{\Sigma}}_{22\cdot 1} = \boldsymbol{S}_{T(12),22} - \boldsymbol{S}_{T(12),21} \boldsymbol{S}_{T(12),11}^{-1} \boldsymbol{S}_{T(12),12}, \quad \widehat{\boldsymbol{\Sigma}}_{33\cdot 12} = \boldsymbol{S}_{(1),33} - \boldsymbol{S}_{(1),3(12)} \boldsymbol{S}_{(1),(12)(12)}^{-1} \boldsymbol{S}_{(1),(12)3}.$$

4.2 Kurtosis test statistic

By approximating a linear interpolation, the expectation $m_{\mathrm{L}}^{(3)}$ and variance $(\nu_{\mathrm{L}}^{(3)})^2$ of $b_{2,p}^{(3)}$ are given as

$$m_{\mathrm{L}}^{(3)} = (p+2)\left(p_1 + \frac{\tau_2}{1-\tau_1} p_2\right) \frac{N-1}{N+1} + (p+2)\left(p_3 + \frac{\tau_3}{1-\tau_1} p_2\right) \frac{N_1 - 1}{N_1 + 1},$$
$$(\nu_{\mathrm{L}}^{(3)})^2 = 8(p+2)\left(p_1 + \frac{\tau_2}{1-\tau_1} p_2\right) \frac{(N-3)(N-p-1)(N-p+1)}{(N+1)^2(N+3)(N+5)}$$
$$+ 8(p+2)\left(p_3 + \frac{\tau_3}{1-\tau_1} p_2\right) \frac{(N_1-3)(N_1-p-1)(N_1-p+1)}{(N_1+1)^2(N_1+3)(N_1+5)}.$$

We note that from Figure 2 that $m_{\mathrm{L}}^{(3)}$ and $(\nu_{\mathrm{L}}^{(3)})^2$ may be between $m_N,\ \nu_N^2$ and $m_{N_1},\ \nu_{N_1}^2$.

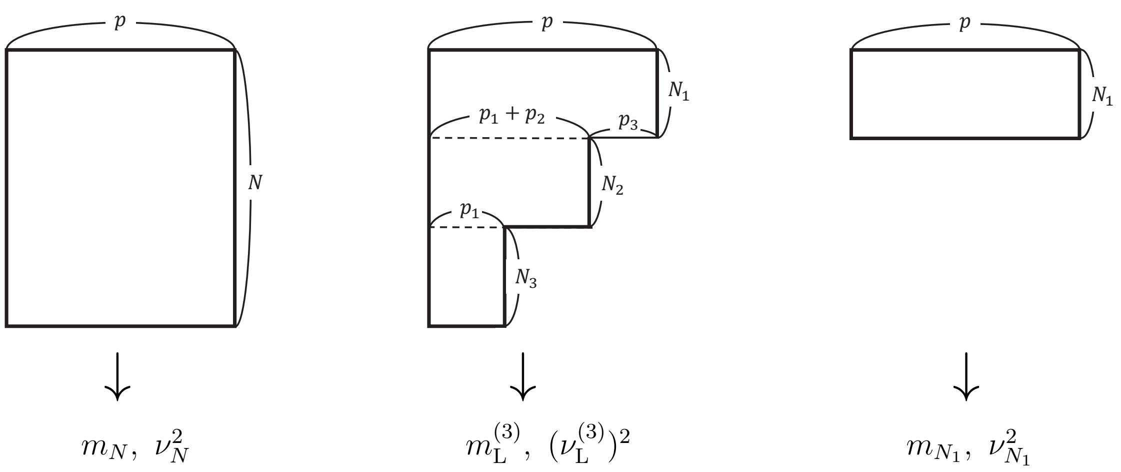

Figure 2. Approximate expectation and variance of multivariate sample kurtosis $b_{2,p}^{(3)}$

Therefore, a new test statistic is given by

$$Z_{\mathrm{MML}}^{(3)} = \frac{b_{2,p}^{(3)} - m_{\mathrm{L}}^{(3)}}{\nu_{\mathrm{L}}^{(3)}},$$

where $Z_{\mathrm{MML}}^{(3)}$ is asymptotically distributed as $N(0,1)$.

5 Simulation study

We investigated the accuracy of the normal approximation for the test statistics $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MML}}^{(3)}$ via Monte Carlo simulation. In the case of two-step monotone missing data, we compared the normal approximation for our test statistic with the corresponding Mardia's type test statistic given by

$$Z_{\mathrm{MM}}^{(2)**} = \frac{b_{2,p}^{(2)} - m_2^{(2)}}{\nu^{(2)}},$$

where

$$m_2^{(2)} = p(p+2) - \frac{2}{N+1}p_1(p_1+2) - \frac{2}{\tau_1 N}p_2(2p_1+p_2+2),$$
$$(\nu^{(2)})^2 = 8p_1(p_1+2)\frac{(N-3)(N-p_1-1)(N-p_1+1)}{(N+1)^2(N+3)(N+5)} + \frac{8}{\tau_1 N}p_2\{(1-\tau_1)p_1p_2 + 2p_1 + p_2 + 2\}.$$

This result can be attributed to Kurita and Seo (2022). Computations were carried out for the following cases:

Case I: Two-step monotone missing data

$(p_1, p_2) = (2,2), (3,3), (5,5), (10,10), (2,4), (4,2), (2,8), (8,2), (16,4), (4,16)$

$N_i = 20, 50, 100, 200, 500, 1000,\ i = 1, 2.$

Case II:Three-step monotone missing data

$(p_1, p_2, p_3) = (2,2,2), (3,3,3), (4,4,4), (5,5,5), (4,2,2), (4,4,2)$

$N_1 = 20, 50, 100, 200, 500, 1000,\ N_2 = N_3 = 10, 20, 50, 100.$

For each parameter, we conducted 10^6 simulated trials based on two- and three-step monotone missing data. For the two-step monotone missing data (Case I), the simulation results related to the expectation and variance of the test statistics $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)**}$, along with their Type I errors for normal approximation, are summarized in tables. Specifically, we selected the case of $(p_1, p_2) = (2,2)$ in Table

1, the case of $(p_1, p_2) = (2, 4)$ in Table 2, and the case of $(p_1, p_2) = (4, 2)$ in Table 3. Furthermore, Tables 1-3 list the simulation results and approximations for $\mathrm{E}[b_{2,p}^{(2)}]$ and $N\mathrm{Var}[b_{2,p}^{(2)}]$, where $m_{\mathrm{L}}^{(2)}$ and $m_2^{(2)}$ are two approximations of $\mathrm{E}[b_{2,p}^{(2)}]$, and $N(\nu_{\mathrm{L}}^{(2)})^2$ and $N(\nu^{(2)})^2$ are two approximations of $N\mathrm{Var}[b_{2,p}^{(2)}]$. In this case, we compared the $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)**}$ tests using their Type I errors defined by

$$\frac{\alpha_{\mathrm{MML}}^{(2)}}{100} = \Pr\left(|Z_{\mathrm{MML}}^{(2)}| > z_{\frac{\alpha}{2}}\right), \ \frac{\alpha_{\mathrm{MM}}^{(2)**}}{100} = \Pr\left(|Z_{\mathrm{MM}}^{(2)**}| > z_{\frac{\alpha}{2}}\right),$$

where $z_{\alpha/2}$ is the upper $100(\alpha/2)$ percentile of the standard normal distribution. It may be noted from Tables 1-3 that $m_{\mathrm{L}}^{(2)}$ and $m_2^{(2)}$ converge to $p(p+2)$ as the sample size N_1 increases. In particular, $m_2^{(2)}$ is closer to the simulated value than to $m_{\mathrm{L}}^{(2)}$ for all cases. Furthermore the simulated value, $m_{\mathrm{L}}^{(2)}$, and $m_2^{(2)}$ are almost the same when the sample size N_1 is larger than the sample size N_2. For variances, Tables 1-3 also show that in all cases, $(\nu_{\mathrm{L}}^{(2)})^2$ is an underestimate while $(\nu^{(2)})^2$ is an overestimate. Furthermore, the simulation results show a tendency for the values of $(\nu_{\mathrm{L}}^{(2)})^2$ to be better approximations when N_2 is small, and for the values of $(\nu^{(2)})^2$ to be better approximations when N_2 is large. For $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)**}$, the tables indicate that as the sample size N_1 increases, their expectation ($M_{\mathrm{MML}}^{(2)}$, $M_{\mathrm{MM}}^{(2)**}$), variance ($(\sigma_{\mathrm{MML}}^{(2)})^2$, $(\sigma_{\mathrm{MM}}^{(2)**})^2$), and empirical Type I error ($\alpha_{\mathrm{MML}}^{(2)}$, $\alpha_{\mathrm{MM}}^{(2)**}$) approach 0,1, and 5, respectively, corresponding to values of the standard normal distribution. The upper and lower percentiles of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)**}$ ($\mathrm{U}_{\mathrm{MML}}^{(2)}$, $\mathrm{L}_{\mathrm{MML}}^{(2)}$ and $\mathrm{U}_{\mathrm{MM}}^{(2)**}$, $\mathrm{L}_{\mathrm{MM}}^{(2)**}$) are also given in Tables 1-3. In particular, the simulation results also show that for each of the test statistics, the Type I error is closer to 5 with the variance closer to 1 than with the expectation closer to 0. In conclusion, the actual Type I error for the test statistic using linear interpolation tends to be close to 5 when the sample size N_1 is large, N_2 is small, and the dimension p_1 exceeds p_2.

Tables list the simulated values for $\mathrm{E}[b_{2,p}^{(3)}]$ and $N\mathrm{Var}[b_{2,p}^{(3)}]$, respectively, along with their approximate values expressed as $m_{\mathrm{L}}^{(3)}$ and $N(\nu_{\mathrm{L}}^{(3)})^2$ for three-step monotone missing data (Case II). They also list results for the expectation($M_{\mathrm{MML}}^{(3)}$), the variance($(\sigma_{\mathrm{MML}}^{(3)})^2$), and the upper and lower percentiles of $Z_{\mathrm{MML}}^{(3)}$($U_{\mathrm{MML}}^{(3)}$, $L_{\mathrm{MML}}^{(3)}$), as well as the actual Type I error, computed as

$$\frac{\alpha_{\mathrm{MML}}^{(3)}}{100} = \Pr\left(|Z_{\mathrm{MML}}^{(3)}| > z_{\frac{\alpha}{2}}\right).$$

Specifically, we selected the case of $(p_1, p_2, p_3) = (2, 2, 2)$ in Table 4, the case of $(p_1, p_2, p_3) = (3, 3, 3)$

in Table 5, and the case of $(p_1, p_2, p_3) = (4, 2, 2)$ in Table 6. It is apparent that approximations of the expectation and variance of $Z^{(3)}_{\mathrm{MML}}$ converge to 0 and 1, respectively, as the sample size N_1 increases. In particular, $\alpha^{(3)}_{\mathrm{MML}}$ can be seen to depend upon $(\sigma^{(3)}_{\mathrm{MML}})^2$; that is, if $(\sigma^{(3)}_{\mathrm{MML}})^2 \leq 1.15$, then $5.0 \leq \alpha^{(3)}_{\mathrm{MML}} < 6.5$. Regarding the null distribution of $Z^{(3)}_{\mathrm{MML}}$ to a normal approximation, when N_2 and N_3 are small while N_1 is large, the values of $\alpha^{(3)}_{\mathrm{MML}}$ are close to 5, indicating that $Z^{(3)}_{\mathrm{MML}}$ has the better approximation. In other words, the proposed $Z^{(3)}_{\mathrm{MML}}$ can be useful when relatively few data are missing. We can observe that this result does not depend upon dimensionality.

6 Concluding Remarks

We employed the test statistics for multivariate kurtosis to test for the multivariate normality with two- and three-step monotone missing data. In particular, we formulated a new test statistic for two-step monotone missing data by using the linear interpolation to the approximate expectation and variance of multivariate sample kurtosis. For the three-step monotone missing data case, multivariate sample kurtosis was defined as an extension of sample kurtosis in the case of two-step monotone missing data as defined by Kurita and Seo (2022). A new statistic was accordingly designed to test for multivariate kurtosis with three-step monotone missing data. Finally, we investigated the accuracy and behavior of the normal approximation by a Monte Carlo simulation. As a result, we found that the proposed test statistics ($Z^{(2)}_{\mathrm{MML}}$ and $Z^{(3)}_{\mathrm{MML}}$) are more accurate when relatively few data are missing.

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Table 1. Expectation and variance of $b_{2,p}^{(2)}$; and expectation, variance, empirical Type I error, and percentiles of test statistics ($Z_{\mathrm{MML}}^{(2)}$, $Z_{\mathrm{MM}}^{(2)**}$), $(p_1, p_2) = (2, 2)$

	Simulation		Approximation											
	$\mathrm{E}[b_{2,p}^{(2)}]$	$N\mathrm{Var}[b_{2,p}^{(2)}]$	$m_{\mathrm{L}}^{(2)}$	$m_2^{(2)}$	$N(\nu_{\mathrm{L}}^{(2)})^2$	$N(\nu^{(2)})^2$	$M_{\mathrm{MML}}^{(2)}$	$M_{\mathrm{MM}}^{(2)**}$	$(\sigma_{\mathrm{MML}}^{(2)})^2$	$(\sigma_{\mathrm{MM}}^{(2)**})^2$	$\alpha_{\mathrm{MML}}^{(2)}$	$\alpha_{\mathrm{MM}}^{(2)**}$	$\begin{pmatrix}\mathrm{U}_{\mathrm{MML}}^{(2)} \\ \mathrm{L}_{\mathrm{MML}}^{(2)}\end{pmatrix}$	$\begin{pmatrix}\mathrm{U}_{\mathrm{MM}}^{(2)**} \\ \mathrm{L}_{\mathrm{MM}}^{(2)**}\end{pmatrix}$
N_1							$N_2 = 20$							
20	22.11	182.92	22.27	22.01	122.21	362.02	-0.10	0.03	1.50	0.51	9.91	1.17	2.66 -2.13	1.63 -1.15
50	23.15	189.53	23.19	23.13	159.15	255.08	-0.03	0.01	1.19	0.74	6.52	2.57	2.42 -1.86	1.94 -1.44
100	23.55	190.04	23.56	23.55	173.90	221.98	-0.01	0.00	1.09	0.86	5.66	3.40	2.28 -1.81	2.03 -1.59
500	23.91	191.67	23.91	23.91	187.93	197.62	0.00	0.00	1.02	0.97	5.15	4.60	2.11 -1.85	2.06 -1.80
1000	23.95	191.83	23.95	23.95	189.93	194.78	0.00	0.00	1.01	0.98	5.04	4.77	2.06 -1.87	2.04 -1.85
							$N_2 = 50$							
20	22.27	351.53	22.52	22.17	185.93	658.28	-0.15	0.03	1.89	0.53	14.59	1.34	2.94 -2.43	1.68 -1.18
50	23.22	282.02	23.29	23.20	203.64	374.04	-0.05	0.01	1.38	0.75	8.74	2.62	2.56 -2.05	1.94 -1.46
100	23.58	242.17	23.60	23.57	200.11	281.16	-0.02	0.00	1.21	0.86	6.95	3.47	2.38 -1.93	2.03 -1.60
500	23.91	203.39	23.91	23.91	193.66	209.25	0.00	0.00	1.05	0.97	5.50	4.64	2.15 -1.87	2.07 -1.80
1000	23.95	197.46	23.95	23.95	192.80	200.57	0.00	0.00	1.02	0.98	5.24	4.80	2.08 -1.89	2.04 -1.85
							$N_2 = 100$							
20	22.38	631.33	22.66	22.27	277.50	1143.58	-0.18	0.04	2.28	0.55	18.78	1.46	3.21 -2.68	1.71 -1.20
50	23.27	434.25	23.37	23.25	272.24	569.16	-0.07	0.01	1.60	0.76	11.19	2.68	2.72 -2.22	1.94 -1.48
100	23.60	328.30	23.64	23.60	241.98	378.80	-0.04	0.00	1.36	0.87	8.65	3.53	2.50 -2.07	2.03 -1.62
500	23.91	221.89	23.91	23.91	203.16	228.61	0.00	0.00	1.09	0.97	5.93	4.58	2.18 -1.92	2.06 -1.80
1000	23.95	207.06	23.95	23.95	197.59	210.22	0.00	0.00	1.05	0.98	5.49	4.79	2.10 -1.91	2.04 -1.85
							$N_2 = 500$							
20	22.48	2834.45	22.81	22.37	945.61	4989.94	-0.24	0.04	3.00	0.57	25.82	1.58	3.67 -3.09	1.74 -1.20
50	23.35	1640.95	23.49	23.33	784.65	2110.05	-0.11	0.01	2.09	0.78	16.78	2.78	3.08 -2.58	1.96 -1.49
100	23.66	1009.34	23.72	23.65	559.38	1150.21	-0.07	0.00	1.80	0.88	13.76	3.63	2.84 -2.42	2.03 -1.64
500	23.92	372.08	23.93	23.92	278.11	382.92	-0.01	0.00	1.34	0.97	8.86	4.64	2.39 -2.15	2.05 -1.82
1000	23.96	282.48	23.96	23.96	235.67	287.28	-0.01	0.00	1.20	0.98	7.26	4.79	2.24 -2.06	2.03 -1.86
							$N_2 = 1000$							
20	22.50	5587.18	22.83	22.38	1768.04	9790.94	-0.25	0.04	3.16	0.57	27.15	1.61	3.77 -3.17	1.75 -1.20
50	23.36	3147.26	23.51	23.34	1415.67	4030.97	-0.13	0.01	2.22	0.78	18.16	2.81	3.17 -2.67	1.96 -1.50
100	23.67	1856.71	23.74	23.67	949.77	2111.02	-0.07	0.00	1.95	0.88	15.44	3.64	2.95 -2.52	2.03 -1.64
500	23.93	559.02	23.94	23.93	370.82	575.28	-0.02	0.00	1.51	0.97	10.86	4.66	2.52 -2.29	2.04 -1.82
1000	23.96	378.61	23.96	23.96	283.01	383.46	-0.01	0.00	1.34	0.99	8.90	4.83	2.35 -2.18	2.03 -1.86

Note. (i) $M_{\mathrm{MML}}^{(2)}$ and $M_{\mathrm{MM}}^{(2)**}$: expectation of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)**}$, (ii) $(\sigma_{\mathrm{MML}}^{(2)})^2$ and $(\sigma_{\mathrm{MM}}^{(2)})^2$: variance of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)**}$, (iii) $\alpha_{\mathrm{MML}}^{(2)}$ and $\alpha_{\mathrm{MM}}^{(2)**}$: empirical Type I error, (iv) $\mathrm{U}_{\mathrm{MML}}^{(2)}$ and $\mathrm{U}_{\mathrm{MM}}^{(2)**}$: upper $100(\alpha/2)$ percentile of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)}$, (v) $L_{\mathrm{MML}}^{(2)}$ and $L_{\mathrm{MM}}^{(2)**}$: lower $100(\alpha/2)$ percentile of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)}$.

Table 2. Expectation and variance of $b_{2,p}^{(2)}$; and expectation, variance, empirical Type I error, and percentiles of test statistics ($Z_{\mathrm{MML}}^{(2)}$, $Z_{\mathrm{MM}}^{(2)**}$), $(p_1, p_2) = (2, 4)$

	Simulation		Approximation											
	$\mathrm{E}[b_{2,p}^{(2)}]$	$N\mathrm{Var}[b_{2,p}^{(2)}]$	$m_{\mathrm{L}}^{(2)}$	$m_2^{(2)}$	$N(\nu_{\mathrm{L}}^{(2)})^2$	$N(\nu^{(2)})^2$	$M_{\mathrm{MML}}^{(2)}$	$M_{\mathrm{MM}}^{(2)**}$	$(\sigma_{\mathrm{MML}}^{(2)})^2$	$(\sigma_{\mathrm{MM}}^{(2)**})^2$	$\alpha_{\mathrm{MML}}^{(2)}$	$\alpha_{\mathrm{MM}}^{(2)**}$	$\begin{pmatrix}\mathrm{U}_{\mathrm{MML}}^{(2)} \\ \mathrm{L}_{\mathrm{MML}}^{(2)}\end{pmatrix}$	$\begin{pmatrix}\mathrm{U}_{\mathrm{MM}}^{(2)**} \\ \mathrm{L}_{\mathrm{MM}}^{(2)**}\end{pmatrix}$
N_1							$N_2 = 20$							
20	43.84	413.09	44.17	43.61	201.14	938.02	-0.15	0.05	2.05	0.44	17.09	0.62	2.94 -2.67	1.48 -1.12
50	46.21	406.96	46.29	46.17	304.02	600.68	-0.04	0.01	1.34	0.68	8.54	1.94	2.46 -2.07	1.79 -1.43
100	47.07	398.31	47.10	47.07	342.38	490.78	-0.02	0.00	1.16	0.81	6.62	3.02	2.30 -1.92	1.94 -1.59
500	47.81	388.42	47.81	47.81	375.21	404.98	0.00	0.00	1.04	0.96	5.34	4.50	2.11 -1.89	2.03 -1.81
1000	47.91	385.57	47.90	47.90	379.57	394.46	0.00	0.00	1.02	0.98	5.15	4.72	2.06 -1.89	2.02 -1.85
							$N_2 = 50$							
20	44.02	892.01	44.50	43.77	323.34	1810.28	-0.22	0.05	2.76	0.49	24.23	0.91	3.38 -3.12	1.57 -1.18
50	46.27	672.85	46.43	46.24	406.50	950.04	-0.08	0.01	1.66	0.71	12.32	2.16	2.69 -2.35	1.82 -1.48
100	47.10	550.47	47.15	47.09	406.52	665.16	-0.03	0.00	1.35	0.83	8.88	3.15	2.45 -2.11	1.94 -1.62
500	47.81	420.60	47.81	47.81	390.14	439.65	0.00	0.00	1.08	0.96	5.84	4.48	2.14 -1.93	2.02 -1.81
1000	47.90	403.55	47.91	47.90	387.14	411.77	0.00	0.00	1.04	0.98	5.45	4.74	2.08 -1.92	2.02 -1.86
							$N_2 = 100$							
20	44.12	1686.47	44.69	43.87	505.35	3255.58	-0.28	0.05	3.34	0.52	29.15	1.09	3.71 -3.44	1.62 -1.20
50	46.33	1113.50	46.53	46.29	568.84	1529.16	-0.10	0.01	1.96	0.73	15.75	2.32	2.90 -2.58	1.84 -1.50
100	47.13	801.26	47.21	47.12	510.61	954.80	-0.05	0.00	1.57	0.84	11.39	3.25	2.61 -2.30	1.95 -1.64
500	47.81	477.20	47.82	47.81	414.96	497.41	-0.01	0.00	1.15	0.96	6.66	4.49	2.20 -2.00	2.02 -1.82
1000	47.91	432.13	47.91	47.91	399.76	440.62	0.00	0.00	1.08	0.98	5.91	4.76	2.12 -1.96	2.02 -1.86
							$N_2 = 500$							
20	44.24	8095.93	44.89	43.97	1862.27	14781.90	-0.35	0.05	4.35	0.55	36.05	1.29	4.23 -3.91	1.67 -1.22
50	46.41	4670.14	46.69	46.37	1811.20	6142.05	-0.15	0.01	2.58	0.76	22.17	2.56	3.28 -3.00	1.88 -1.54
100	47.18	2817.35	47.31	47.17	1315.93	3262.21	-0.09	0.00	2.14	0.86	17.82	3.52	3.00 -2.74	1.97 -1.68
500	47.82	924.49	47.84	47.82	611.82	958.92	-0.02	0.00	1.51	0.96	10.98	4.58	2.49 -2.33	2.01 -1.84
1000	47.91	657.90	47.91	47.91	500.39	671.28	-0.01	0.00	1.31	0.98	8.68	4.74	2.31 -2.18	2.01 -1.87
							$N_2 = 1000$							
20	44.26	16089.80	44.92	43.98	3538.53	29182.90	-0.36	0.05	4.55	0.55	37.15	1.31	4.32 -4.01	1.68 -1.22
50	46.43	9091.76	46.71	46.38	3349.24	11903.00	-0.16	0.01	2.71	0.76	23.44	2.60	3.37 -3.08	1.89 -1.54
100	47.20	5332.54	47.34	47.19	2312.59	6143.02	-0.10	0.01	2.31	0.87	19.51	3.53	3.11 -2.84	1.97 -1.68
500	47.83	1484.79	47.85	47.83	856.36	1535.28	-0.03	0.00	1.73	0.97	13.55	4.61	2.66 -2.50	2.01 -1.85
1000	47.91	942.17	47.92	47.91	625.76	959.46	-0.01	0.00	1.51	0.98	10.96	4.77	2.47 -2.34	2.00 -1.88

Note. (i) $M_{\mathrm{MML}}^{(2)}$ and $M_{\mathrm{MM}}^{(2)**}$: expectation of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)**}$, (ii) $(\sigma_{\mathrm{MML}}^{(2)})^2$ and $(\sigma_{\mathrm{MM}}^{(2)})^2$: variance of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)**}$, (iii) $\alpha_{\mathrm{MML}}^{(2)}$ and $\alpha_{\mathrm{MM}}^{(2)**}$: empirical Type I error, (iv) $\mathrm{U}_{\mathrm{MML}}^{(2)}$ and $\mathrm{U}_{\mathrm{MM}}^{(2)**}$: upper $100(\alpha/2)$ percentile of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)}$, (v) $L_{\mathrm{MML}}^{(2)}$ and $L_{\mathrm{MM}}^{(2)**}$: lower $100(\alpha/2)$ percentile of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)}$.

Table 3. Expectation and variance of $b_{2,p}^{(2)}$; and expectation, variance, empirical Type I error, and percentiles of test statistics ($Z_{\mathrm{MML}}^{(2)}$, $Z_{\mathrm{MM}}^{(2)**}$), $(p_1, p_2) = (4, 2)$

	Simulation		Approximation											
	$\mathrm{E}[b_{2,p}^{(2)}]$	$N\mathrm{Var}[b_{2,p}^{(2)}]$	$m_{\mathrm{L}}^{(2)}$	$m_2^{(2)}$	$N(\nu_{\mathrm{L}}^{(2)})^2$	$N(\nu^{(2)})^2$	$M_{\mathrm{MML}}^{(2)}$	$M_{\mathrm{MM}}^{(2)**}$	$(\sigma_{\mathrm{MML}}^{(2)})^2$	$(\sigma_{\mathrm{MM}}^{(2)**})^2$	$\alpha_{\mathrm{MML}}^{(2)}$	$\alpha_{\mathrm{MM}}^{(2)**}$	$\begin{pmatrix}\mathrm{U}_{\mathrm{MML}}^{(2)} \\ \mathrm{L}_{\mathrm{MML}}^{(2)}\end{pmatrix}$	$\begin{pmatrix}\mathrm{U}_{\mathrm{MM}}^{(2)**} \\ \mathrm{L}_{\mathrm{MM}}^{(2)**}\end{pmatrix}$
N_1							$N_2 = 20$							
20	44.58	318.61	44.92	44.43	201.47	625.13	-0.15	0.04	1.58	0.51	11.49	1.05	2.61 -2.31	1.60 -1.19
50	46.39	344.57	46.47	46.36	285.62	462.09	-0.04	0.01	1.21	0.75	6.94	2.50	2.36 -1.94	1.90 -1.48
100	47.13	359.11	47.15	47.12	326.81	417.13	-0.02	0.00	1.10	0.86	5.85	3.43	2.24 -1.86	2.00 -1.63
500	47.81	378.25	47.81	47.81	370.58	389.20	0.00	0.00	1.02	0.97	5.17	4.64	2.09 -1.87	2.04 -1.82
1000	47.91	380.57	47.91	47.91	377.13	386.49	0.00	0.00	1.01	0.98	5.07	4.81	2.05 -1.89	2.03 -1.86
							$N_2 = 50$							
20	45.10	626.98	45.57	44.92	295.28	1134.09	-0.23	0.04	2.12	0.55	18.14	1.28	2.96 -2.74	1.67 -1.24
50	46.59	506.83	46.74	46.56	352.40	667.56	-0.08	0.01	1.44	0.76	9.71	2.58	2.52 -2.18	1.90 -1.51
100	47.21	449.62	47.26	47.20	365.60	518.89	-0.03	0.00	1.23	0.87	7.35	3.50	2.35 -2.00	2.00 -1.65
500	47.82	396.79	47.82	47.82	378.53	408.81	0.00	0.00	1.05	0.97	5.49	4.62	2.11 -1.90	2.04 -1.83
1000	47.91	390.13	47.91	47.91	381.05	396.20	0.00	0.00	1.02	0.98	5.24	4.81	2.07 -1.90	2.03 -1.86
							$N_2 = 100$							
20	45.39	1125.18	45.95	45.20	408.29	1953.13	-0.30	0.05	2.76	0.58	24.75	1.43	3.33 -3.17	1.71 -1.26
50	46.75	772.44	46.95	46.72	446.76	998.89	-0.12	0.01	1.73	0.77	13.18	2.70	2.73 -2.43	1.91 -1.54
100	47.29	596.40	47.36	47.28	424.64	684.85	-0.05	0.00	1.40	0.87	9.47	3.55	2.48 -2.17	1.99 -1.66
500	47.82	428.76	47.83	47.82	391.63	441.39	-0.01	0.00	1.09	0.97	6.01	4.63	2.15 -1.95	2.03 -1.83
1000	47.91	406.26	47.91	47.91	387.56	412.37	0.00	0.00	1.05	0.99	5.51	4.79	2.09 -1.92	2.03 -1.86
							$N_2 = 500$							
20	45.69	5018.32	46.35	45.51	1114.10	8376.40	-0.45	0.05	4.50	0.60	37.67	1.59	4.21 -4.09	1.75 -1.28
50	46.98	2830.76	47.26	46.95	1089.05	3576.81	-0.19	0.01	2.60	0.79	22.49	2.84	3.28 -3.04	1.93 -1.56
100	47.45	1745.74	47.58	47.44	842.12	1977.39	-0.11	0.00	2.07	0.88	17.11	3.69	2.94 -2.71	2.00 -1.69
500	47.86	680.53	47.87	47.86	493.17	700.01	-0.02	0.00	1.38	0.97	9.42	4.66	2.38 -2.22	2.02 -1.84
1000	47.92	533.79	47.93	47.92	439.02	541.33	-0.01	0.00	1.22	0.99	7.51	4.83	2.23 -2.09	2.02 -1.88
							$N_2 = 1000$							
20	45.74	9860.55	46.41	45.55	1956.61	16380.10	-0.49	0.05	5.04	0.60	40.59	1.63	4.45 -4.32	1.75 -1.28
50	47.02	5384.66	47.31	46.99	1862.10	6780.20	-0.22	0.01	2.36	0.79	25.16	2.86	3.44 -3.22	1.93 -1.56
100	47.48	3173.68	47.63	47.48	1343.98	3580.37	-0.13	0.00	2.36	0.89	20.08	3.72	3.13 -2.90	2.00 -1.69
500	47.87	994.46	47.89	47.87	617.00	1021.33	-0.03	0.00	1.61	0.97	12.15	4.69	2.57 -2.41	2.02 -1.85
1000	47.93	691.62	47.94	47.93	502.49	701.99	-0.02	0.00	1.38	0.99	9.42	4.81	2.36 -2.24	2.01 -1.88

Note. (i) $M_{\mathrm{MML}}^{(2)}$ and $M_{\mathrm{MM}}^{(2)**}$: expectation of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)**}$, (ii) $(\sigma_{\mathrm{MML}}^{(2)})^2$ and $(\sigma_{\mathrm{MM}}^{(2)})^2$: variance of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)**}$, (iii) $\alpha_{\mathrm{MML}}^{(2)}$ and $\alpha_{\mathrm{MM}}^{(2)**}$: empirical Type I error, (iv) $\mathrm{U}_{\mathrm{MML}}^{(2)}$ and $\mathrm{U}_{\mathrm{MM}}^{(2)**}$: upper $100(\alpha/2)$ percentile of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)}$, (v) $L_{\mathrm{MML}}^{(2)}$ and $L_{\mathrm{MM}}^{(2)**}$: lower $100(\alpha/2)$ percentile of $Z_{\mathrm{MML}}^{(2)}$ and $Z_{\mathrm{MM}}^{(2)}$.

Table 4. Expectation and variance of $b_{2,p}^{(3)}$; and expectation, variance, empirical Type I error, and percentiles of $Z_{\text{MML}}^{(3)}$, $(p_1, p_2, p_3) = (2, 2, 2)$

	Simulation		Approximation		$Z_{\text{MML}}^{(3)}$				
	$\text{E}[b_{2,p}^{(3)}]$	$N\text{Var}[b_{2,p}^{(3)}]$	$m_{\text{L}}^{(3)}$	$N(\nu_{\text{L}}^{(3)})^2$	$M_{\text{MML}}^{(3)}$	$(\sigma_{\text{MML}}^{(3)})^2$	$\alpha_{\text{MML}}^{(3)}$	$\text{L}_{\text{MML}}^{(3)}$	$\text{U}_{\text{MML}}^{(3)}$
N_1				$N_2 = N_3 = 10$					
20	44.33	352.21	44.54	201.30	-0.10	1.75	13.42	-2.40	2.77
50	46.31	372.27	46.38	294.82	-0.03	1.26	7.60	-1.99	2.41
100	47.11	378.51	47.13	334.59	-0.01	1.13	6.24	-1.89	2.28
200	47.54	380.79	47.54	357.56	-0.01	1.06	5.59	-1.87	2.18
500	47.81	383.09	47.81	372.90	0.00	1.03	5.23	-1.88	2.09
1000	47.91	382.58	47.91	378.35	0.00	1.01	5.12	-1.89	2.06
				$N_2 = N_3 = 20$					
20	44.72	586.67	44.93	276.22	-0.10	2.12	17.35	-2.63	3.07
50	46.43	501.39	46.53	352.24	-0.05	1.42	9.54	-2.14	2.53
100	47.14	452.84	47.18	369.20	-0.02	1.23	7.34	-1.99	2.35
200	47.55	422.01	47.56	376.14	-0.01	1.12	6.24	-1.93	2.23
500	47.81	399.74	47.82	380.53	0.00	1.05	5.51	-1.90	2.12
1000	47.91	392.36	47.91	382.18	0.00	1.02	0.053	-1.90	2.07
				$N_2 = N_3 = 50$					
20	45.19	1224.59	45.32	456.82	-0.07	2.68	22.50	-2.92	3.49
50	46.65	865.31	46.74	507.80	-0.05	1.70	12.89	-2.36	2.76
100	47.24	667.30	47.29	467.63	-0.03	1.43	9.73	-2.17	2.51
200	47.58	541.42	47.60	430.60	-0.02	1.26	7.85	-2.06	2.34
500	47.82	450.49	47.82	403.29	-0.01	1.12	6.30	-1.97	2.18
1000	47.91	418.21	47.91	393.66	-0.00	1.06	0.057	-1.94	2.10
				$N_2 = N_3 = 100$					
20	45.44	2239.19	45.50	723.50	-0.03	3.09	25.70	-3.08	3.81
50	46.80	1420.88	46.87	749.42	-0.04	1.90	14.90	-2.48	2.92
100	47.31	994.27	47.37	624.04	-0.04	1.59	11.69	-2.31	2.63
200	47.62	726.39	47.64	519.01	-0.02	1.40	9.52	-2.19	2.45
500	47.83	532.15	47.84	440.92	-0.01	1.21	7.34	-2.06	2.25
1000	47.91	461.91	47.91	412.74	-0.01	1.12	0.063	-2.00	2.14

Note. (i) $M_{\text{MML}}^{(3)}$: expectation of $Z_{\text{MML}}^{(3)}$, (ii) $(\sigma_{\text{MML}}^{(3)})^2$: variance of $Z_{\text{MML}}^{(3)}$, (iii) $\alpha_{\text{MML}}^{(3)}$: empirical Type I error, (iv) $U_{\text{MML}}^{(3)}$: upper $100(\alpha/2)$ percentile of $Z_{\text{MML}}^{(3)}$, (v) $L_{\text{MML}}^{(3)}$: lower $100(\alpha/2)$ percentile of $Z_{\text{MML}}^{(3)}$.

Table 5. Expectation and variance of $b_{2,p}^{(3)}$; and expectation, variance, empirical Type I error, and percentiles of $Z_{\mathrm{MML}}^{(3)}$, $(p_1, p_2, p_3) = (3, 3, 3)$

	Simulation		Approximation		$Z_{\mathrm{MML}}^{(3)}$				
	$\mathrm{E}[b_{2,p}^{(3)}]$	$N\mathrm{Var}[b_{2,p}^{(3)}]$	$m_{\mathrm{L}}^{(3)}$	$N(\nu_{\mathrm{L}}^{(3)})^2$	$M_{\mathrm{MML}}^{(3)}$	$(\sigma_{\mathrm{MML}}^{(3)})^2$	$\alpha_{\mathrm{MML}}^{(3)}$	$\mathrm{L}_{\mathrm{MML}}^{(3)}$	$\mathrm{U}_{\mathrm{MML}}^{(3)}$
N_1				$N_2 = N_3 = 10$					
20	91.33	749.40	91.87	300.41	-0.20	2.49	21.87	-3.04	3.23
50	95.50	773.88	95.66	539.01	-0.06	1.44	9.88	-2.21	2.48
100	97.15	784.36	97.20	650.24	-0.02	1.21	7.20	-2.00	2.30
200	98.04	788.18	98.06	715.84	-0.01	1.10	6.06	-1.93	2.19
500	98.61	790.63	98.61	759.96	0.00	1.04	5.42	-1.91	2.09
1000	98.80	793.06	98.80	775.68	0.00	1.02	5.23	-1.92	2.05
				$N_2 = N_3 = 20$					
20	92.15	1358.93	92.66	422.24	-0.19	3.22	27.89	-3.43	3.59
50	95.75	1112.78	95.97	649.18	-0.08	1.71	13.28	-2.45	2.67
100	97.23	981.42	97.32	719.77	-0.04	1.36	9.08	-2.15	2.42
200	98.07	896.07	98.10	753.83	-0.02	1.19	7.08	-2.02	2.25
500	98.61	833.38	98.62	775.68	0.00	1.07	5.81	-1.94	2.12
1000	98.81	813.77	98.81	783.59	0.00	1.04	5.38	-1.93	2.06
				$N_2 = N_3 = 50$					
20	93.10	3074.12	93.47	686.59	-0.15	4.48	35.62	-3.95	4.32
50	96.17	2061.26	96.40	939.64	-0.09	2.19	18.46	-2.79	3.01
100	97.41	1540.14	97.53	915.11	-0.06	1.68	12.91	-2.44	2.65
200	98.13	1206.40	98.18	864.65	-0.03	1.40	9.55	-2.22	2.41
500	98.63	972.00	98.64	822.50	-0.01	1.18	7.09	-2.05	2.21
1000	98.81	885.00	98.81	807.25	0.00	1.10	6.10	-1.99	2.12
				$N_2 = N_3 = 100$					
20	93.62	5779.95	93.84	1044.32	-0.10	5.53	40.43	-4.33	4.89
50	96.46	3532.06	96.66	1380.43	-0.09	2.56	21.91	-3.01	3.26
100	97.56	2399.26	97.69	1221.71	-0.06	1.96	16.02	-2.65	2.84
200	98.20	1690.45	98.26	1043.38	-0.04	1.62	12.25	-2.41	2.58
500	98.65	1188.24	98.66	899.72	-0.01	1.32	8.74	-2.18	2.33
1000	98.81	996.55	98.82	846.55	-0.01	1.18	7.05	-2.07	2.19

Note. (i) $M_{\mathrm{MML}}^{(3)}$: expectation of $Z_{\mathrm{MML}}^{(3)}$, (ii) $(\sigma_{\mathrm{MML}}^{(3)})^2$: variance of $Z_{\mathrm{MML}}^{(3)}$, (iii) $\alpha_{\mathrm{MML}}^{(3)}$: empirical Type I error, (iv) $U_{\mathrm{MML}}^{(3)}$: upper $100(\alpha/2)$ percentile of $Z_{\mathrm{MML}}^{(3)}$, (v) $L_{\mathrm{MML}}^{(3)}$: lower $100(\alpha/2)$ percentile of $Z_{\mathrm{MML}}^{(3)}$.

Table 6. Expectation and variance of $b_{2,p}^{(3)}$; and expectation, variance, empirical Type I error, and percentiles of $Z_{\mathrm{MML}}^{(3)}$, $(p_1, p_2, p_3) = (4, 2, 2)$

	Simulation		Approximation		$Z_{\mathrm{MML}}^{(3)}$				
	$\mathrm{E}[b_{2,p}^{(3)}]$	$N\mathrm{Var}[b_{2,p}^{(3)}]$	$m_{\mathrm{L}}^{(3)}$	$N(\nu_{\mathrm{L}}^{(3)})^2$	$M_{\mathrm{MML}}^{(3)}$	$(\sigma_{\mathrm{MML}}^{(3)})^2$	$\alpha_{\mathrm{MML}}^{(3)}$	$\mathrm{L}_{\mathrm{MML}}^{(3)}$	$\mathrm{U}_{\mathrm{MML}}^{(3)}$
N_1				$N_2 = N_3 = 10$					
20	74.30	531.42	74.70	278.22	-0.15	1.91	15.60	-2.59	2.81
50	77.29	576.38	77.42	444.82	-0.05	1.30	8.14	-2.07	2.39
100	78.54	603.53	78.58	527.23	-0.02	1.14	6.47	-1.93	2.26
200	79.24	620.79	79.25	578.27	-0.01	1.07	5.70	-1.89	2.17
500	79.69	632.26	79.69	613.72	0.00	1.03	5.29	-1.89	2.09
1000	79.84	633.55	79.84	626.56	-0.00	1.01	5.12	-1.90	2.04
				$N_2 = N_3 = 20$					
20	75.08	898.24	75.50	380.72	-0.17	2.36	20.35	-2.89	3.12
50	77.54	783.20	77.72	523.87	-0.07	1.50	10.56	-2.25	2.54
100	78.63	723.85	78.70	574.27	-0.03	1.26	7.81	-2.05	2.35
200	79.27	685.85	79.29	602.94	-0.01	1.14	6.46	-1.96	2.22
500	79.69	660.18	79.70	623.56	0.00	1.06	5.62	-1.92	2.12
1000	79.84	648.59	79.84	631.44	-0.00	1.03	5.31	-1.92	2.06
				$N_2 = N_3 = 50$					
20	75.97	1874.92	76.32	589.03	-0.15	3.18	27.23	-3.33	3.66
50	77.97	1338.11	78.16	722.32	-0.09	1.85	14.70	-2.54	2.80
100	78.81	1052.39	78.91	702.71	-0.05	1.50	10.67	-2.27	2.52
200	79.33	868.61	79.37	673.92	-0.03	1.29	8.26	-2.11	2.33
500	79.71	738.87	79.71	652.79	-0.01	1.13	6.47	-2.00	2.17
1000	79.85	691.55	79.85	646.02	-0.00	1.07	5.77	-1.96	2.10
				$N_2 = N_3 = 100$					
20	76.44	3356.73	76.69	856.40	-0.13	3.92	32.16	-3.63	4.12
50	78.26	2155.48	78.43	1011.61	-0.08	2.13	17.74	-2.72	2.99
100	78.97	1540.31	79.07	898.68	-0.06	1.71	13.25	-2.45	2.68
200	79.40	1149.31	79.45	786.56	-0.03	1.46	10.32	-2.27	2.47
500	79.72	865.09	79.74	700.73	-0.01	1.23	7.69	-2.10	2.25
1000	79.85	755.75	79.86	670.19	-0.01	1.13	6.48	-2.02	2.15

Note. (i) $M_{\mathrm{MML}}^{(3)}$: expectation of $Z_{\mathrm{MML}}^{(3)}$, (ii) $(\sigma_{\mathrm{MML}}^{(3)})^2$: variance of $Z_{\mathrm{MML}}^{(3)}$, (iii) $\alpha_{\mathrm{MML}}^{(3)}$: empirical Type I error, (iv) $U_{\mathrm{MML}}^{(3)}$: upper $100(\alpha/2)$ percentile of $Z_{\mathrm{MML}}^{(3)}$, (v) $L_{\mathrm{MML}}^{(3)}$: lower $100(\alpha/2)$ percentile of $Z_{\mathrm{MML}}^{(3)}$.