On the Extension of Test Statistics for the Sub-mean Vector under Two-step Monotone Missing Data

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Abstract

In this paper, we consider the problem of testing for the sub-mean vector when the data have a two-step monotone missing pattern. In the case where the data set consists of complete data with $p(= p_1+p_2+p_3+p_4)$ dimensions and incomplete data with p_1+p_2 dimensions, we consider the one-sample problem of testing the $p_2 + p_3 + p_4$ mean vector, $p_3 + p_4$ mean vector, and p_4 mean vector under the given mean vector of remaining dimensions. Moreover, we investigate the accuracy using Monte Carlo simulation.

1 Introduction

We consider the one-sample problem of testing for a sub-mean vector. Rao (1949) discussed this problem for non-missing data, and Siotani, Hayakawa and Fujikoshi (1985) introduced Rao's U-statistic and the derivation of its null distribution. For two-step monotone missing data, the problem of testing for a sub-mean vector is discussed in Kawasaki and Seo (2016a), where the likelihood ratio test statistic and its approximate upper percentiles are proposed. In considering the hypothesis of a sub-mean vector with two-step monotone missing data, as an alternative approach, we construct a test statistic based on Rao's U-statistic structure.

First, we consider two-step monotone missing data. Let $\boldsymbol{x}_1^{(1)}, \boldsymbol{x}_2^{(1)}, \dots, \boldsymbol{x}_{N_1}^{(1)}$ be distributed as the multivariate normal $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and $\boldsymbol{x}_1^{(2)}, \boldsymbol{x}_2^{(2)}, \dots, \boldsymbol{x}_{N_2}^{(2)}$ be distributed as the multivariate normal $N_{p_1+p_2}(\boldsymbol{\mu}_{(12)}, \boldsymbol{\Sigma}_{(12)(12)})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_3 \\ \boldsymbol{\mu}_4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_{(234)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)} \\ \boldsymbol{\mu}_{(34)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(123)} \\ \boldsymbol{\mu}_4 \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \boldsymbol{\Sigma}_{13} & \boldsymbol{\Sigma}_{14} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \boldsymbol{\Sigma}_{23} & \boldsymbol{\Sigma}_{24} \\ \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33} & \boldsymbol{\Sigma}_{34} \\ \boldsymbol{\Sigma}_{41} & \boldsymbol{\Sigma}_{42} & \boldsymbol{\Sigma}_{43} & \boldsymbol{\Sigma}_{44} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{(12)(12)} & \boldsymbol{\Sigma}_{(12)3} & \boldsymbol{\Sigma}_{(12)4} \\ \boldsymbol{\Sigma}_{3(12)} & \boldsymbol{\Sigma}_{33} & \boldsymbol{\Sigma}_{34} \\ \boldsymbol{\Sigma}_{4(12)} & \boldsymbol{\Sigma}_{43} & \boldsymbol{\Sigma}_{44} \end{pmatrix}.$$

We partition $\boldsymbol{x}_j^{(1)}$ into a $p_1 \times 1$ random vector, a $p_2 \times 1$ random vector, a $p_3 \times 1$ random vector, and a $p_4 \times 1$ random vector with $\boldsymbol{x}_j^{(1)} = (\boldsymbol{x}_{1j}^{(1)\prime}, \boldsymbol{x}_{2j}^{(1)\prime}, \boldsymbol{x}_{3j}^{(1)\prime}, \boldsymbol{x}_{4j}^{(1)\prime})'$, where $\boldsymbol{x}_{ij}^{(1)} : p_i \times 1,\ i = 1, 2, 3, 4,\ j = 1, 2, \ldots, N_1$, and partition $\boldsymbol{x}_j^{(2)}$ into a $p_1 \times 1$ random vector and a $p_2 \times 1$ random vector with $\boldsymbol{x}_j^{(2)} = (\boldsymbol{x}_{1j}^{(2)\prime}, \boldsymbol{x}_{2j}^{(2)\prime})'$, where $\boldsymbol{x}_{ij}^{(2)} : p_i \times 1,\ i = 1, 2,\ j = 1, 2, \ldots, N_2$. Then, the two-step monotone missing data can be written as

$$\begin{pmatrix} \overbrace{\boldsymbol{x}_{11}^{(1)\prime}}^{p_1} & \overbrace{\boldsymbol{x}_{21}^{(1)\prime}}^{p_2} & \overbrace{\boldsymbol{x}_{31}^{(1)\prime}}^{p_3} & \overbrace{\boldsymbol{x}_{41}^{(1)\prime}}^{p_4} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{x}_{1N_1}^{(1)\prime} & \boldsymbol{x}_{2N_1}^{(1)\prime} & \boldsymbol{x}_{3N_1}^{(1)\prime} & \boldsymbol{x}_{4N_1}^{(1)\prime} \\ \boldsymbol{x}_{11}^{(2)\prime} & \boldsymbol{x}_{21}^{(2)\prime} & * \cdots * & * \cdots * \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \quad \vdots \\ \boldsymbol{x}_{1N_2}^{(2)\prime} & \boldsymbol{x}_{2N_2}^{(2)\prime} & * \cdots * & * \cdots * \end{pmatrix},$$

where $N = N_1 + N_2$, $p = p_1 + p_2 + p_3 + p_4$ and '$*$' indicates missing data.

For the case of two-step monotone missing data, tests for a sub-mean vector can be classified into the following three patterns:

$$H_0 : \boldsymbol{\mu}_{(234)} = \boldsymbol{\mu}_{(234)0} \text{ given } \boldsymbol{\mu}_1 = \boldsymbol{\mu}_{10} \text{ vs. } H_1 : \boldsymbol{\mu}_{(234)} \neq \boldsymbol{\mu}_{(234)0} \text{ given } \boldsymbol{\mu}_1 = \boldsymbol{\mu}_{10}, \tag{1}$$

$$H_0' : \boldsymbol{\mu}_{(34)} = \boldsymbol{\mu}_{(34)0} \text{ given } \boldsymbol{\mu}_{(12)} = \boldsymbol{\mu}_{(12)0} \text{ vs. } H_1' : \boldsymbol{\mu}_{(34)} \neq \boldsymbol{\mu}_{(34)0} \text{ given } \boldsymbol{\mu}_{(12)} = \boldsymbol{\mu}_{(12)0}, \tag{2}$$

$$H_0'' : \boldsymbol{\mu}_4 = \boldsymbol{\mu}_{40} \text{ given } \boldsymbol{\mu}_{(123)} = \boldsymbol{\mu}_{(123)0} \text{ vs. } H_1'' : \boldsymbol{\mu}_4 \neq \boldsymbol{\mu}_{40} \text{ given } \boldsymbol{\mu}_{(123)} = \boldsymbol{\mu}_{(123)0}, \tag{3}$$

where $\boldsymbol{\mu}_0$ partitioned and $\boldsymbol{\mu}$ are known.

As a test statistic for these hypotheses, we propose Rao's U-type statistic based on Rao's U-statistic structure. When composing Rao's U-type statistic, we use Hotelling's T^2-type test statistic. Hotelling's T^2-type test statistic and its approximate upper 100α percentiles were proposed by Seko, Yamazaki and Seo (2012). Furthermore, Kawasaki and Seo (2016b) derived an expansion of the T^2-type test statistic and provided a modified Bartlett corrected statistic.

Anderson (1957) developed an approach to derive the MLEs for monotone missing data. Kanda and Fujikoshi (1998) discussed the distribution of MLEs in the cases of general k-step monotone missing data. Srivastava and Carter (1986) used the Newton–Raphson method to obtain the MLEs of the mean vector and covariance matrix for general-type missing data.

In section 2, we consider the test statistic of hypotheses (1), (2) and (3). Specifically, we determine the distribution of Rao's U-type statistics and approximate upper 100α percentiles.

In section 3, we perform Monte Carlo simulations and discuss the results. We show the usefulness of the approximate upper percentiles of the test statistic by means of a numerical example test in section 4. Finally, in section 5, we conclude our article. The proofs of some results are provided in the Appendix.

2 Rao's U-type statistic

We propose a test statistic for the sub-mean vector with two-step monotone missing data. First, we consider hypotheses (1) and (2). Since hypothesis (2) is essentially the same as hypothesis (1) with $p_2 = 0$, we consider hypothesis (1).

In hypothesis (1), the test statistic is given by

$$U_{p_1} = (N-p)\frac{T^2_{Mp} - T^2_{p_1}}{N-1+T^2_{p_1}},$$

where

$$T^2_{Mp} = (\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0)'\{\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}})\}^{-1}(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0),$$

$$T^2_{p_1} = N(\overline{\boldsymbol{x}}_{1T} - \boldsymbol{\mu}_{10})'\boldsymbol{S}_{1T}^{-1}(\overline{\boldsymbol{x}}_{1T} - \boldsymbol{\mu}_{10}),$$

$$\overline{\boldsymbol{x}}_{1T} = \frac{1}{N}(N_1\overline{\boldsymbol{x}}_1^{(1)} + N_2\overline{\boldsymbol{x}}_1^{(2)}),$$

$$\boldsymbol{S}_{1T} = \frac{1}{N-1}\left\{\sum_{j=1}^{N_1}(\boldsymbol{x}_{1j}^{(1)} - \overline{\boldsymbol{x}}_{1T})(\boldsymbol{x}_{1j}^{(1)} - \overline{\boldsymbol{x}}_{1T})' + \sum_{j=1}^{N_2}(\boldsymbol{x}_{1j}^{(2)} - \overline{\boldsymbol{x}}_{1T})(\boldsymbol{x}_{1j}^{(2)} - \overline{\boldsymbol{x}}_{1T})'\right\},$$

$$\overline{\boldsymbol{x}}_1^{(1)} = \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{x}_{1j}^{(1)},\ \overline{\boldsymbol{x}}_2^{(1)} = \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{x}_{2j}^{(1)},\ \overline{\boldsymbol{x}}_3^{(1)} = \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{x}_{3j}^{(1)},\ \overline{\boldsymbol{x}}_4^{(1)} = \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{x}_{4j}^{(1)},$$

$$\overline{\boldsymbol{x}}_1^{(2)} = \frac{1}{N_2}\sum_{j=1}^{N_2}\boldsymbol{x}_{1j}^{(2)},\ \overline{\boldsymbol{x}}_2^{(2)} = \frac{1}{N_2}\sum_{j=1}^{N_2}\boldsymbol{x}_{2j}^{(2)}.$$

Here, T^2_{Mp} is the Hotelling's T^2-type statistic. It was obtained by Seko, Yamazaki and Seo (2012). $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$ are the MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ respectively, and $\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}})$ is the estimator obtained using the respective MLEs for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in $\mathrm{Cov}(\widehat{\boldsymbol{\mu}}), (N_1 > p_1 + p_2 + 2)$ (Kanda and Fujikoshi (1998)). Furthermore, $\overline{\boldsymbol{x}}^{(1)} = (\overline{\boldsymbol{x}}_1^{(1)'}, \overline{\boldsymbol{x}}_2^{(1)'}, \overline{\boldsymbol{x}}_3^{(1)'}, \overline{\boldsymbol{x}}_4^{(1)'})'$, $\overline{\boldsymbol{x}}^{(2)} = (\overline{\boldsymbol{x}}_1^{(2)'}, \overline{\boldsymbol{x}}_2^{(2)'})'$, and the sample covariance matricies are defined as

$$\boldsymbol{S}^{(1)} = \frac{1}{n_1}\sum_{j=1}^{N_1}(\boldsymbol{x}_j^{(1)} - \overline{\boldsymbol{x}}^{(1)})(\boldsymbol{x}_j^{(1)} - \overline{\boldsymbol{x}}^{(1)})', \boldsymbol{S}^{(2)} = \frac{1}{n_2}\sum_{j=1}^{N_2}(\boldsymbol{x}_j^{(2)} - \overline{\boldsymbol{x}}^{(2)})(\boldsymbol{x}_j^{(2)} - \overline{\boldsymbol{x}}^{(2)})',$$

where $n_1 = N_1 - 1, n_2 = N_2 - 1$. We call this test statistic the Rao's U-type statistic.

For convenience, let $\boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}$ and $\boldsymbol{V}^{(1)}, \boldsymbol{V}^{(2)}$ be

$$\boldsymbol{z}^{(1)} = \sqrt{N_1}(\overline{\boldsymbol{x}}^{(1)} - \boldsymbol{\mu}),\ \boldsymbol{z}^{(2)} = \sqrt{N_2}(\overline{\boldsymbol{x}}^{(2)} - \boldsymbol{\mu}_{(12)}),$$

$$\boldsymbol{V}^{(1)} = \sqrt{n_1}(\boldsymbol{S}^{(1)} - \boldsymbol{\Sigma}),\ \boldsymbol{V}^{(2)} = \sqrt{n_2}(\boldsymbol{S}^{(2)} - \boldsymbol{\Sigma}_{(12)(12)}).$$

The probability expansion under $n = n_1 + n_2,\ \gamma_i = n_i/n \to \delta_i \in (0,1)$ and $n_1, n_2 \to \infty$ yields the following formula, where $\boldsymbol{\mu} = \boldsymbol{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}$ to avoid loss of generality. Using the results of the probability expansions of T^2_{Mp} and $T^2_{p_1}$, the expansion of U_{p_1} can be obtained.

$$U_{p_1} = Q_0 + \frac{1}{\sqrt{n}}Q_1 + \frac{1}{n}Q_2 + \frac{1}{n\sqrt{n}}Q_3 + O_p(n^{-2}),$$

where

$$\begin{aligned}
Q_0 =& \mathrm{tr}\boldsymbol{\Lambda}_{22} + \boldsymbol{z}^{(1)\prime}_{(34)}\boldsymbol{z}^{(1)}_{(34)},\\
Q_1 =& -\frac{1}{\sqrt{\gamma_1}}(2\boldsymbol{z}^{(1)\prime}_{(12)}\boldsymbol{V}^{(1)}_{(12)(34)}\boldsymbol{z}^{(1)}_{(34)} + \boldsymbol{z}^{(1)\prime}_{(34)}\boldsymbol{V}^{(1)}_{(34)(34)}\boldsymbol{z}^{(1)}_{(34)})\\
&- \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\boldsymbol{\Lambda}\} + \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{11} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)}_{11})\boldsymbol{\Lambda}_{11}\},\\
Q_2 =& \mathrm{tr}\boldsymbol{\Lambda}_{11} + \boldsymbol{z}^{(1)\prime}_{2}\boldsymbol{z}^{(1)}_{2} + \boldsymbol{z}^{(2)\prime}_{2}\boldsymbol{z}^{(2)}_{2} + \frac{1}{\sqrt{\gamma_1\gamma_2}}\boldsymbol{z}^{(1)\prime}_{2}\boldsymbol{z}^{(2)}_{2}\\
&+ \left\{\frac{1}{\gamma_1}(1 - p_1 - p_2) + (1 - p_3 - p_4)\right\}\boldsymbol{z}^{(1)\prime}_{(34)}\boldsymbol{z}^{(1)}_{(34)}\\
&- \boldsymbol{z}^{(1)\prime}_{(34)}\boldsymbol{z}^{(1)}_{(34)}\mathrm{tr}\boldsymbol{\Lambda}_{11} + \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})^2\boldsymbol{\Lambda}\} - \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{11} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)}_{11})^2\boldsymbol{\Lambda}_{11}\}\\
&- \gamma_1\gamma_2\mathrm{tr}\left\{\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'\boldsymbol{\Lambda}\right\} + (1 - p - \mathrm{tr}\boldsymbol{\Lambda}_{11})\mathrm{tr}\boldsymbol{\Lambda}_{22}\\
&+ \frac{1}{\gamma_1}\{\boldsymbol{z}^{(1)\prime}_{(12)}\boldsymbol{V}^{(1)}_{(12)(34)}\boldsymbol{V}^{(1)}_{(34)(12)}\boldsymbol{z}^{(1)}_{(12)} + 2\boldsymbol{z}^{(1)\prime}_{(12)}\boldsymbol{V}^{(1)}_{(12)(12)}\boldsymbol{V}^{(1)}_{(12)(34)}\boldsymbol{z}^{(1)}_{(34)} + 2\boldsymbol{z}^{(1)\prime}_{(12)}\boldsymbol{V}^{(1)}_{(12)(34)}\boldsymbol{V}^{(1)}_{(34)(34)}\boldsymbol{z}^{(1)}_{(34)}\\
&+ \boldsymbol{z}^{(1)\prime}_{(34)}\boldsymbol{V}^{(1)}_{(34)(12)}\boldsymbol{V}^{(1)}_{(12)(34)}\boldsymbol{z}^{(1)}_{(34)} + \boldsymbol{z}^{(1)\prime}_{(34)}(\boldsymbol{V}^{(1)}_{(34)(34)})^2\boldsymbol{z}^{(1)}_{(34)}\},\\
\boldsymbol{\Lambda} =& (\sqrt{\gamma_1}\boldsymbol{z}^{(1)}_{(12)} + \sqrt{\gamma_2}\boldsymbol{z}^{(2)})(\sqrt{\gamma_1}\boldsymbol{z}^{(1)}_{(12)} + \sqrt{\gamma_2}\boldsymbol{z}^{(2)})' = \begin{pmatrix}\boldsymbol{\Lambda}_{11} & \boldsymbol{\Lambda}_{12}\\ \boldsymbol{\Lambda}_{21} & \boldsymbol{\Lambda}_{22}\end{pmatrix}.
\end{aligned}$$

We omit Q_3 owing to space limitations. We partition $\boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}, \boldsymbol{V}^{(1)}$, and $\boldsymbol{V}^{(2)}$ as

$$\boldsymbol{z}^{(1)} = \begin{pmatrix}\boldsymbol{z}^{(1)}_1\\ \boldsymbol{z}^{(1)}_2\\ \boldsymbol{z}^{(1)}_3\\ \boldsymbol{z}^{(1)}_4\end{pmatrix} = \begin{pmatrix}\boldsymbol{z}^{(1)}_1\\ \boldsymbol{z}^{(1)}_2\\ \boldsymbol{z}^{(1)}_{(34)}\end{pmatrix}, \boldsymbol{z}^{(2)} = \begin{pmatrix}\boldsymbol{z}^{(2)}_1\\ \boldsymbol{z}^{(2)}_2\end{pmatrix},$$

$$\boldsymbol{V}^{(1)} = \begin{pmatrix}\boldsymbol{V}^{(1)}_{11} & \boldsymbol{V}^{(1)}_{12} & \boldsymbol{V}^{(1)}_{13} & \boldsymbol{V}^{(1)}_{14}\\ \boldsymbol{V}^{(1)}_{21} & \boldsymbol{V}^{(1)}_{22} & \boldsymbol{V}^{(1)}_{23} & \boldsymbol{V}^{(1)}_{24}\\ \boldsymbol{V}^{(1)}_{31} & \boldsymbol{V}^{(1)}_{32} & \boldsymbol{V}^{(1)}_{33} & \boldsymbol{V}^{(1)}_{34}\\ \boldsymbol{V}^{(1)}_{41} & \boldsymbol{V}^{(1)}_{42} & \boldsymbol{V}^{(1)}_{43} & \boldsymbol{V}^{(1)}_{44}\end{pmatrix} = \begin{pmatrix}\boldsymbol{V}^{(1)}_{(12)(12)} & \boldsymbol{V}^{(1)}_{(12)(34)}\\ \boldsymbol{V}^{(1)}_{(34)(12)} & \boldsymbol{V}^{(1)}_{(34)(34)}\end{pmatrix}, \boldsymbol{V}^{(2)} = \begin{pmatrix}\boldsymbol{V}^{(2)}_{11} & \boldsymbol{V}^{(2)}_{12}\\ \boldsymbol{V}^{(2)}_{21} & \boldsymbol{V}^{(2)}_{22}\end{pmatrix},$$

and $\boldsymbol{V}^{(1)}_{ij}, \boldsymbol{V}^{(2)}_{ij}$ are $p_i \times p_j$ matrices. We also denote $p_{(12)} = p_1 + p_2,\ p_{(34)} = p_3 + p_4$ and $\boldsymbol{V}^{(1)}_{(ij)(kl)}$ is a $p_{(ij)} \times p_{(kl)}$ matrix. The process of deriving U_{p_1} is given in detail in the Appendix. From these probability expansion results, we propose the following theorem.

Theorem 1.

Let U_{p_1} be the test statistic for hypothesis (1) and (2). The distribution function of U_{p_1} is given by

$$\Pr(U_{p_1} \leq x) = G_{p-p_1}(x) + \frac{1}{n}\sum_{j=0}^{2} \beta_j G_{p-p_1+2j}(x) + O(n^{-2}),$$

where

$$\begin{aligned}
\beta_0 =& -p_2 - \frac{1+\gamma_1}{2\gamma_1}p_{(34)} - \frac{1-\gamma_1}{2\gamma_1}p_1 p_{(34)} + \frac{1}{4}p_2^2 - \frac{1-2\gamma_1}{2\gamma_1}p_2 p_{(34)} - \frac{1-2\gamma_1}{4\gamma_1}p_{(34)}^2,\\
\beta_1 =& \frac{1}{2}p_2 + \frac{1}{2}p_{(34)} + \frac{1-\gamma_1}{2\gamma_1}p_1 p_{(34)} - \frac{1}{2}p_2^2 + \frac{1-3\gamma_1}{2\gamma_1}p_2 p_{(34)} - \frac{1}{2}p_{(34)}^2,\\
\beta_2 =& \frac{1}{2}p_2 + \frac{1}{2\gamma_1}p_{(34)} + \frac{1}{4}p_2^2 + \frac{1}{2}p_2 p_{(34)} + \frac{1}{4\gamma_1}p_{(34)}^2.
\end{aligned}$$

$G_f(x)$ is the distribution function of a chi-squared distribution with f degrees of freedom. The approximate upper 100α percentiles u_{p_1} can be obtained as.

$$u_{p_1}(\alpha) = \chi^2_{p-p_1}(\alpha) + \frac{2}{n}\chi^2_{p-p_1}(\alpha)\left\{\frac{\beta_2\chi^2_{p-p_1}(\alpha)}{(p-p_1)(p-p_1+2)} + \frac{\beta_1+\beta_2}{p-p_1}\right\} + O(n^{-2}),$$

where $\chi^2_f(\alpha)$ is the upper 100α percentiles of the chi-squared distribution with f degrees of freedom.

With this result, the approximate upper 100α percentiles $u^*_{p_1}(\alpha)$ can be proposed.

$$u^*_{p_1}(\alpha) = \chi^2_{p-p_1}(\alpha) + \frac{2}{n}\chi^2_{p-p_1}(\alpha)\left\{\frac{\beta_2\chi^2_{p-p_1}(\alpha)}{(p-p_1)(p-p_1+2)} + \frac{\beta_1+\beta_2}{p-p_1}\right\}.$$

Furthermore, from the distribution,

$$E[U_{p_1}] = p - p_1 + \frac{1}{n}(2\beta_1 + 4\beta_2) + O(n^{-2}).$$

Then, the Bartlett correction can be obtained as

$$\widetilde{U}_{p_1} = \left\{1 - \frac{2\beta_1+4\beta_2}{n(p-p_1)}\right\}U_{p_1}, \qquad \left(n > \frac{2\beta_1+4\beta_2}{p-p_1}\right).$$

Next, in the hypothesis (3), the test statistic is given by

$$U_{p_{(123)}} = (N-p)\frac{T^2_{Mp} - T^2_{Mp_{(123)}}}{N-1+T^2_{Mp_{(123)}}},$$

where

$$T^2_{Mp} = (\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0)'\{\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}})\}^{-1}(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0),$$

$$T^2_{Mp_{(123)}} = (\widehat{\boldsymbol{\mu}}_{(123)} - \boldsymbol{\mu}_{(123)0})'\{\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}}_{(123)})\}^{-1}(\widehat{\boldsymbol{\mu}}_{(123)} - \boldsymbol{\mu}_{(123)0}).$$

Here, T^2_{Mp} and $T^2_{Mp_{(123)}}$ are Hotelling's T^2 type statistics. We do the same probability expansion for $U_{p_{(123)}}$ as for U_{p_1}.

$$U_{p_{(123)}} = R_0 + \frac{1}{\sqrt{n}}R_1 + \frac{1}{n}R_2 + \frac{1}{n\sqrt{n}}R_3 + O_p(n^{-2}),$$

$$\begin{aligned}
R_0 =& \boldsymbol{z}_4^{(1)\prime}\boldsymbol{z}_4^{(1)},\\
R_1 =& -\frac{1}{\sqrt{\gamma_1}}(2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)4}^{(1)}\boldsymbol{z}_4^{(1)} + \boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} - \boldsymbol{z}_3^{(1)\prime}\boldsymbol{V}_{33}^{(1)}\boldsymbol{z}_3^{(1)}),\\
R_2 =& \left\{(1-p) - \frac{1}{\gamma_1}(\gamma_2 p_{(12)} + 1)\right\}\boldsymbol{z}_4^{(1)\prime}\boldsymbol{z}_4^{(1)} - \boldsymbol{z}_4^{(1)\prime}\boldsymbol{z}_4^{(1)}(\mathrm{tr}\boldsymbol{\Lambda}_{(12)(12)} + \boldsymbol{z}_3^{(1)\prime}\boldsymbol{z}_3^{(1)})\\
&+ \frac{1}{\gamma_1}\{\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{z}_{(12)}^{(1)} + 2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)}\\
&+ 2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + \boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + \boldsymbol{z}_{(34)}^{(1)\prime}(\boldsymbol{V}_{(34)(34)}^{(1)})^2\boldsymbol{z}_{(34)}^{(1)}\}\\
&- \frac{1}{\gamma_1}\{\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)3}^{(1)}\boldsymbol{V}_{3(12)}^{(1)}\boldsymbol{z}_{(12)}^{(1)} + 2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(12)}^{(1)}\boldsymbol{V}_{(12)3}^{(1)}\boldsymbol{z}_3^{(1)}\\
&+ 2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)3}^{(1)}\boldsymbol{V}_{33}^{(1)}\boldsymbol{z}_3^{(1)} + \boldsymbol{z}_3^{(1)\prime}\boldsymbol{V}_{3(12)}^{(1)}\boldsymbol{V}_{(12)3}^{(1)}\boldsymbol{z}_3^{(1)} + \boldsymbol{z}_3^{(1)\prime}(\boldsymbol{V}_{33}^{(1)})^2\boldsymbol{z}_3^{(1)}\}.
\end{aligned}$$

R_3 is omitted owing to space limitations. The process of deriving $U_{p_{(123)}}$ is given in detail in the Appendix. From this probability expansion results, we propose the following theorem.

Theorem 2.

Let $U_{p_{(123)}}$ be the test statistic for hypothesis (3). The distribution function of $U_{p_{(123)}}$ is given by

$$\Pr(U_{p_{(123)}} \le x) = G_{p_4}(x) + \frac{1}{n}\sum_{j=0}^{2}\zeta_j G_{p_4+2j}(x) + O(n^{-2}),$$

where

$$\begin{aligned}
\zeta_0 =& -\frac{1+\gamma_1}{2\gamma_1}p_4 - \frac{1-\gamma_1}{2\gamma_1}p_{(12)}p_4 - \frac{1-\gamma_1}{\gamma_1}p_3p_4 - \frac{1-2\gamma_1}{4\gamma_1}p_4^2,\\
\zeta_1 =& \frac{1}{2}p_4 + \frac{1-\gamma_1}{2\gamma_1}p_{(12)}p_4 + \frac{1-\gamma_1}{\gamma_1}p_3p_4 - \frac{1}{2}p_4^2,\\
\zeta_2 =& \frac{1}{2\gamma_1}p_4 + \frac{1}{4\gamma_1}p_4^2.
\end{aligned}$$

We obtained the approximate upper 100α percentiles $u_{p_{(123)}}$.

$$\begin{aligned}
u_{p_{(123)}}(\alpha) =& \chi^2_{p_4}(\alpha) + \frac{2}{n}\chi^2_{p_4}(\alpha)\left\{\frac{\zeta_2\chi^2_{p_4}(\alpha)}{p_4(p_4+2)} + \frac{\zeta_1+\zeta_2}{p_4}\right\} + O(n^{-2})\\
=& \chi^2_{p_4}(\alpha)\\
&+ \frac{1}{n}\chi^2_{p_4}(\alpha)\left\{\frac{1}{2\gamma_1}\chi^2_{p_4}(\alpha) + \frac{1-\gamma_1}{\gamma_1}p_{(12)} + \frac{1}{\gamma_1}p_3 + \frac{1-2\gamma_1}{2\gamma_1}p_4 + \frac{2-\gamma_1}{\gamma_1}\right\} + O(n^{-2}).
\end{aligned}$$

Then, the approximate upper 100α percentiles of $U^*_{p_{(123)}}$ is as follows

$$u^*_{p_{(123)}}(\alpha) = \chi^2_{p_4}(\alpha) + \frac{1}{n}\chi^2_{p_4}(\alpha)\left\{\frac{1}{2\gamma_1}\chi^2_{p_4}(\alpha) + \frac{1-\gamma_1}{\gamma_1}p_{(12)} + \frac{1}{\gamma_1}p_3 + \frac{1-2\gamma_1}{2\gamma_1}p_4 + \frac{2-\gamma_1}{\gamma_1}\right\}.$$

Similar to that of $\widetilde{U}_{p_1}$, the Bartlett correction of $U_{p_{(123)}}$ can be obtained as follows.

$$\widetilde{U}_{p_{(123)}} = \left\{1 - \frac{1}{n}\Big(\frac{2+\gamma_1}{\gamma_1} + \frac{1-\gamma_1}{\gamma_1}p_{(12)} + \frac{2-2\gamma_1}{\gamma_1}p_3 + \frac{1-\gamma_1}{\gamma_1}p_4\Big)\right\}U_{p_{(123)}},$$
$$\left(n > \frac{2+\gamma_1}{\gamma_1} + \frac{1-\gamma_1}{\gamma_1}p_{(12)} + \frac{2-2\gamma_1}{\gamma_1}p_3 + \frac{1-\gamma_1}{\gamma_1}p_4\right).$$

3 Simulation

In this section, we perform Monte Carlo simulations (with 10^6 runs) in order to verify the approximation accuracies of $u^*_{p_1}$ and $u^*_{p_{(123)}}$ and the Bartlett corrections $\widetilde{U}_{p_1}$ and $\widetilde{U}_{p_{(123)}}$. We compute the 100α percentiles of U_{p_1}, $U_{p_1}(\alpha)$ and the 100α percentiles of $\widetilde{U}_{p_1}$, $\widetilde{u}_{p_1}(\alpha)$. We generate artificial two-step missing monotone data from $N_p(\mathbf{0}, \boldsymbol{I}_p)$ for various conditions of p_1, p_2, p_3, p_4, N_1, and N_2. We define

$$P_{chi_1} = \Pr\{U_{p_1} > \chi^2_{p-p_1}(\alpha)\},\ P_{u_1} = \Pr\{U_{p_1} > u^*_{p_1}(\alpha)\},$$
$$P_{b_1} = \Pr\{\widetilde{U}_{p_1} > \chi^2_{p-p_1}(\alpha)\},$$

and $\chi^2_f(\alpha)$ is the upper 100α percentile of the χ^2 distribution with f degrees of freedom. We present the simulation results for the following tables. First, we set the dimensionality $p_1 = p_2 = p_3 + p_4 = 2$ and the equal sample sizes $N_1 = N_2 = 20, 40, 80, 160, 320, 640$. In Table 1, we set the significance level as $\alpha = 0.01$ and 0.05. Furthermore, we perform the simulation for α fixed as 0.05. For Table 2, we set the dimensionality $(p_1, p_2, p_3 + p_4) = (4, 2, 2), (2, 4, 2), (2, 2, 4)$. Furthermore, Table 3 shows the case where the sample size ratio in Table 2 is changed to $N_1 : N_2 = 1 : 2, 2 : 1$.

Table 1: $(p_1, p_2, p_3 + p_4) = (2, 2, 2)$, and $\alpha = 0.05, 0.01$

	N_1	N_2	$U_{p_1}(\alpha)$	$u^*_{p_1}(\alpha)$	$\widetilde{u}_{p_1}(\alpha)$	P_{chi_1}	P_{u_1}	P_{b_1}
$\alpha = 0.05$								
	20	20	13.16	11.82	10.75	0.119	0.068	0.071
	40	40	10.75	10.62	9.79	0.076	0.052	0.056
	80	80	10.02	10.05	9.58	0.061	0.050	0.052
	160	160	9.73	9.77	9.52	0.055	0.049	0.051
	320	320	9.61	9.63	9.50	0.052	0.050	0.050
	640	640	9.54	9.56	9.50	0.051	0.050	0.050
$\alpha = 0.01$								
	20	20	21.13	17.42	17.25	0.049	0.020	0.026
	40	40	15.76	15.29	14.36	0.022	0.012	0.015
	80	80	14.30	14.27	13.68	0.015	0.010	0.012
	160	160	13.75	13.77	13.45	0.012	0.010	0.011
	320	320	13.50	13.52	13.36	0.011	0.010	0.010
	640	640	13.38	13.40	13.31	0.010	0.010	0.010

Note: $\chi^2_4(0.05) = 9.49$, $\chi^2_4(0.01) = 13.28$

Table 2: p is fixed, and $\alpha = 0.05$

p_1	p_2	$p_3 + p_4$	N_1	N_2	$U_{p_1}(\alpha)$	$u^*_{p_1}(\alpha)$	$\widetilde{u}_{p_1}(\alpha)$	P_{chi_1}	P_{u_1}	P_{b_1}
4	2	2	20	20	14.68	12.07	11.59	0.146	0.084	0.085
$\chi^2_4(0.05) = 9.49$			40	40	11.03	10.74	9.90	0.082	0.055	0.058
			80	80	10.09	10.11	9.58	0.062	0.050	0.052
			160	160	9.74	9.80	9.50	0.055	0.049	0.050
			320	320	9.63	9.64	9.50	0.053	0.050	0.050
			640	640	9.55	9.56	9.49	0.051	0.050	0.050
2	4	2	20	20	18.09	15.58	15.08	0.139	0.079	0.086
$\chi^2_6(0.05) = 12.59$			40	40	14.16	14.05	13.01	0.078	0.052	0.057
			80	80	13.17	13.31	12.64	0.061	0.048	0.051
			160	160	12.84	12.95	12.59	0.055	0.048	0.050
			320	320	12.69	12.77	12.56	0.052	0.049	0.050
			640	640	12.65	12.68	12.59	0.051	0.050	0.050
2	2	4	20	20	21.84	16.94	-	0.199	0.101	-
$\chi^2_6(0.05) = 12.59$			40	40	15.38	14.71	13.47	0.102	0.059	0.065
			80	80	13.73	13.64	12.89	0.071	0.051	0.055
			160	160	13.08	13.11	12.68	0.059	0.050	0.052
			320	320	12.83	12.85	12.64	0.054	0.050	0.051
			640	640	12.72	12.72	12.62	0.052	0.050	0.050

Table 3: p is fixed, and $\alpha = 0.05$

p_1	p_2	$p_3 + p_4$	N_1	N_2	$U_{p_1}(\alpha)$	$u^*_{p_1}(\alpha)$	$\widetilde{u}_{p_1}(\alpha)$	P_{chi_1}	P_{u_1}	P_{b_1}
4	2	2	10	20	155.10	15.09	-	0.473	0.346	-
$\chi^2_4(0.05) = 9.49$			20	40	15.00	12.14	11.57	0.152	0.087	0.084
			40	80	11.10	10.78	9.87	0.084	0.056	0.057
			80	160	10.13	10.12	9.57	0.064	0.050	0.052
			160	320	9.79	9.80	9.52	0.056	0.050	0.051
			320	640	9.61	9.65	9.48	0.052	0.049	0.050
			20	10	14.34	11.99	11.59	0.138	0.080	0.084
			40	20	10.93	10.71	9.90	0.079	0.054	0.058
			80	40	10.04	10.09	9.58	0.061	0.049	0.052
			160	80	9.74	9.79	9.51	0.055	0.049	0.050
			320	160	9.60	9.64	9.49	0.052	0.049	0.050
			640	320	9.55	9.56	9.49	0.051	0.050	0.050
2	4	2	10	20	168.54	18.50	-	0.460	0.341	-
$\chi^2_6(0.05) = 12.59$			20	40	18.13	15.40	15.05	0.141	0.082	0.086
			40	80	14.19	13.96	13.02	0.079	0.053	0.057
			80	160	13.21	13.27	12.68	0.061	0.049	0.051
			160	320	12.87	12.93	12.61	0.055	0.049	0.050
			320	640	12.73	12.76	12.60	0.053	0.050	0.050
			20	10	18.14	15.76	15.17	0.137	0.076	0.087
			40	20	14.16	14.13	13.03	0.077	0.050	0.057
			80	40	13.18	13.35	12.66	0.060	0.047	0.051
			160	80	12.82	12.97	12.57	0.054	0.048	0.050
			320	160	12.69	12.78	12.56	0.052	0.048	0.049
			640	320	12.66	12.69	12.60	0.051	0.050	0.050
2	2	4	10	20	324.01	22.39	-	0.649	0.469	-
$\chi^2_6(0.05) = 12.59$			20	40	22.81	17.22	-	0.218	0.108	-
			40	80	15.64	14.85	13.46	0.108	0.061	0.064
			80	160	13.84	13.70	12.89	0.074	0.052	0.055
			160	320	13.17	13.14	12.72	0.061	0.050	0.052
			320	640	12.88	12.87	12.66	0.055	0.050	0.051
			20	10	20.86	16.64	-	0.181	0.094	-
			40	20	15.11	14.57	13.48	0.096	0.057	0.064
			80	40	13.63	13.57	12.90	0.069	0.051	0.055
			160	80	13.06	13.08	12.71	0.059	0.050	0.052
			320	160	12.81	12.83	12.64	0.054	0.050	0.051
			640	320	12.70	12.71	12.61	0.052	0.050	0.050

Next, we define

$$P_{chi_{(123)}} = \Pr\{U_{p_{(123)}} > \chi^2_{p_4}(\alpha)\},\ P_{u_{(123)}} = \Pr\{U_{p_{(123)}} > u^*_{p_{(123)}}(\alpha)\},$$
$$P_{b_{(123)}} = \Pr\{\widetilde{U}_{p_{(123)}} > \chi^2_{p_4}(\alpha)\}.$$

We set the dimensionality $p_1 + p_2 = p_3 = p_4 = 2$ and the equal sample sizes $N_1 = N_2 = 20, 40, 80, 160, 320, 640$. We compute the 100α percentiles of $U_{p_{(123)}}$, $U_{p_{(123)}}(\alpha)$ and the 100α percentiles of $\widetilde{U}_{p_{(123)}}$, $\widetilde{u}_{p_{(123)}}(\alpha)$. In Table 4, we set the significance level as $\alpha = 0.01$ and 0.05. For Table 5, we set the dimensionality $(p_1+p_2, p_3, p_4) = (4, 2, 2), (2, 4, 2), (2, 2, 4)$, where α fixed as 0.05. Finally, Table 6 shows the case where the sample size ratio in Table 5 is changed to $N_1 : N_2 = 1 : 2, 2 : 1$.

Table 4: $(p_1 + p_2, p_3, p_4) = (2, 2, 2)$, and $\alpha = 0.05, 0.01$

	N_1	N_2	$U_{p_{(123)}}(\alpha)$	$u^*_{p_{(123)}}(\alpha)$	$\widetilde{u}_{p_{(123)}}(\alpha)$	$P_{chi_{(123)}}$	$P_{u_{(123)}}$	$P_{b_{(123)}}$
$\alpha = 0.05$								
	20	20	9.95	8.36	6.55	0.141	0.074	0.061
	40	40	7.43	7.14	6.19	0.086	0.056	0.055
	80	80	6.63	6.56	6.08	0.066	0.052	0.052
	160	160	6.28	6.27	6.02	0.057	0.050	0.051
	320	320	6.13	6.13	6.01	0.053	0.050	0.050
	640	640	6.06	6.06	6.00	0.052	0.050	0.050
$\alpha = 0.01$								
	20	20	17.52	13.62	11.53	0.060	0.022	0.020
	40	40	12.04	11.36	10.03	0.026	0.013	0.014
	80	80	10.44	10.27	9.58	0.017	0.011	0.012
	160	160	9.77	9.74	9.37	0.013	0.010	0.011
	320	320	9.49	9.47	9.30	0.011	0.010	0.010
	640	640	9.34	9.34	9.25	0.011	0.010	0.010

Note: $\chi^2_2(0.05) = 5.99$, $\chi^2_2(0.01) = 9.21$

Table 5: p is fixed, and $\alpha = 0.05$

$p_1 + p_2$	p_3	p_4	N_1	N_2	$U_{p_{(123)}}(\alpha)$	$u^*_{p_{(123)}}(\alpha)$	$\widetilde{u}_{p_{(123)}}(\alpha)$	$P_{chi_{(123)}}$	$P_{u_{(123)}}$	$P_{b_{(123)}}$
4	2	2	20	20	11.54	8.67	6.98	0.173	0.092	0.070
$\chi^2_2(0.05) = 5.99$			40	40	7.78	7.30	6.29	0.095	0.059	0.057
			80	80	6.75	6.64	6.11	0.069	0.052	0.053
			160	160	6.33	6.31	6.03	0.058	0.050	0.051
			320	320	6.15	6.15	6.01	0.054	0.050	0.050
			640	640	6.08	6.07	6.01	0.052	0.050	0.050
2	4	2	20	20	12.06	8.99	6.67	0.186	0.093	0.063
$\chi^2_2(0.05) = 5.99$			40	40	7.94	7.45	6.21	0.099	0.059	0.055
			80	80	6.81	6.71	6.08	0.071	0.052	0.052
			160	160	6.40	6.35	6.06	0.060	0.051	0.052
			320	320	6.17	6.17	6.01	0.055	0.050	0.051
			640	640	6.07	6.08	5.99	0.052	0.050	0.050
2	2	4	20	20	19.09	14.10	-	0.229	0.106	-
$\chi^2_4(0.05) = 9.49$			40	40	12.51	11.74	10.10	0.114	0.062	0.062
			80	80	10.75	10.60	9.73	0.077	0.053	0.055
			160	160	10.06	10.04	9.59	0.062	0.050	0.052
			320	320	9.75	9.76	9.52	0.055	0.050	0.051
			640	640	9.61	9.62	9.50	0.053	0.050	0.050

Table 6: p is fixed, and $\alpha = 0.05$

$p_1 + p_2$	p_3	p_4	N_1	N_2	$U_{p_{(123)}}(\alpha)$	$u^*_{p_{(123)}}(\alpha)$	$\widetilde{u}_{p_{(123)}}(\alpha)$	$P_{chi_{(123)}}$	$P_{u_{(123)}}$	$P_{b_{(123)}}$
4	2	2	10	20	189.39	12.48	-	0.602	0.428	-
$\chi^2_2(0.05) = 5.99$			20	40	12.74	9.05	6.67	0.198	0.102	0.064
			40	80	8.10	7.48	6.23	0.103	0.061	0.055
			80	160	6.87	6.73	6.09	0.072	0.053	0.052
			160	320	6.38	6.36	6.02	0.060	0.051	0.051
			320	640	6.20	6.17	6.02	0.055	0.051	0.051
			20	10	10.38	8.28	7.16	0.147	0.082	0.074
			40	20	7.42	7.11	6.29	0.085	0.056	0.057
			80	40	6.61	6.55	6.11	0.066	0.051	0.053
			160	80	6.28	6.27	6.04	0.057	0.050	0.051
			320	160	6.14	6.13	6.02	0.054	0.050	0.051
			640	320	6.07	6.06	6.01	0.052	0.050	0.050
2	4	2	10	20	222.46	12.91	-	0.652	0.468	-
$\chi^2_2(0.05) = 5.99$			20	40	13.67	9.26	-	0.219	0.112	-
			40	80	8.37	7.58	6.15	0.111	0.065	0.054
			80	160	6.99	6.78	6.08	0.075	0.055	0.052
			160	320	6.46	6.38	6.04	0.062	0.052	0.051
			320	640	6.22	6.19	6.01	0.056	0.051	0.051
			20	10	10.62	8.70	6.97	0.153	0.077	0.070
			40	20	7.54	7.32	6.26	0.088	0.054	0.056
			80	40	6.63	6.65	6.08	0.066	0.050	0.052
			160	80	6.29	6.32	6.03	0.057	0.049	0.051
			320	160	6.12	6.15	6.00	0.053	0.049	0.050
			640	320	6.07	6.07	6.01	0.052	0.050	0.050
2	2	4	10	20	356.51	20.55	-	0.732	0.515	-
$\chi^2_4(0.05) = 9.49$			20	40	21.09	14.71	-	0.266	0.120	-
			40	80	13.08	12.03	10.05	0.128	0.066	0.060
			80	160	10.97	10.74	9.72	0.082	0.054	0.054
			160	320	10.17	10.11	9.59	0.065	0.051	0.052
			320	640	9.83	9.80	9.55	0.057	0.051	0.051
			20	10	17.18	13.48	-	0.193	0.093	-
			40	20	12.02	11.44	10.19	0.102	0.059	0.063
			80	40	10.55	10.46	9.76	0.072	0.052	0.055
			160	80	9.97	9.97	9.59	0.060	0.050	0.052
			320	160	9.72	9.73	9.54	0.055	0.050	0.051
			640	320	9.61	9.61	9.52	0.053	0.050	0.051

From Tables 1 and 2, we can see that the proposed $u^*_{p_1}$ provides a good result in the case that the sample sizes n_1 and n_2 are large. Furthermore, the larger the dimension, the worse the approximation accuracy. Specifically, for the same dimension, the worst approximation is when $p_3 + p_4$ is large, and the best approximation is when p_2 is large. Table 3 shows that $u^*_{p_1}$ produces the best results for different ratios of dimensions as well.

From Tables 4 and 5, we can see that the Bartlett correction $\widetilde{U}_{p_{(123)}}$ is the best approximation when the dimension is small, and the proposed $u^*_{p_{(123)}}$ is the best when the dimension is large. Table 6 shows that when $N_1 : N_2 = 1 : 2$, the Bartlett correction $\widetilde{U}_{p_{(123)}}$ results are better, and when $N_1 : N_2 = 2 : 1$, the $u^*_{p_{(123)}}$ results are better.

4 Numerical example

In this section, we investigate the approximate upper 100α percentiles and the Bartlett correction evaluated via a Monte Carlo simulation. We illustrate using crude oil production for each U.S. region over the five-year period from 2018 to 2022[1]. The U.S. Energy Information Administration (EIA) publishes the data for 32 states (Alabama, Alaska, Arizona, Arkansas, California, Colorado, Florida, Idaho, Illinois, Indiana , Kansas, Kentucky, Louisiana, Michigan, Mississippi, Missouri, Montana, Nebraska, Nevada, New Mexico, New York, North Dakota, Ohio, Oklahoma, Pennsylvania, South Dakota, Tennessee, Texas, Utah, Virginia, West Virginia and Wyoming). We create two-step monotone missing data by deleting the values for 12 states for 2022. Therefore, we have $N = 32, N_1 = 20, N_2 = 12, p = 5, p_1 = 2, p_2 = 2$ and $p_3 = 1$. We are interested in whether the impact of the COVID-19 pandemic after 2020 would have changed U.S. crude oil production. The pre-pandemic mean is stable, with a mean of 86699 for both 2018 and 2019, when compared to the sample mean for 2015–2017. We consider the hypothesis

$$H : (\mu_3, \mu_4, \mu_5)' = (86699, 86699, 86699)' \text{ given } (\mu_1, \mu_2)' = (86699, 86699)'$$
$$\text{vs. } H' : (\mu_3, \mu_4, \mu_5)' \neq (86699, 86699, 86699)' \text{ given } (\mu_1, \mu_2)' = (86699, 86699)'.$$

Then, we compute $U_{p_1} = 13.19$ and $\widetilde{U}_{p_1} = 11.28$. From $\chi^2_{p-p_1}(0.05) = 7.82$ and $u^*_{p_1}(0.05) = 9.36$, we do not reject the null hypothesis at the 0.05 signicance level. In addition, the null hypothesis at the 0.01 signicance level is rejected when U_{p_1} and $\chi^2_{p-p_1}(0.01) = 11.34$ are used but not when U_{p_1} and $u^*_{p_1}(0.01) = 14.32$ or $\widetilde{U}_{p_1}$ and $\chi^2_{p-p_1}(0.01)$ are used.

5 Concluding remarks

In this paper, we considered the one-sample problem of testing for a sub-mean vector with two-step monotone missing data. First, we proposed Rao's U-type statistic and obtained a stochastic expansion. We derived the approximate upper 100α percentiles and the Bartlett correction using a stochastic expansion of Rao's U-type statistics. Finaly, we performed Monte

[1] `https://www.eia.gov/dnav/pet/pet_crd_crpdn_adc_mbbl_a.htm`, Annual crude oil production by U.S., PAD District and state, accessed February 29, 2024.

Carlo simulations in order to verify the approximation accuracy. The results revealed that the values of $u^*_{p_1}(\alpha)$ and $u^*_{p_{(123)}}(\alpha)$ were satisfactory.

Appendix

(A.1.)

The probability expansion of T^2_{Mp} and $T^2_{p_1}$ can be given as follows.

$$T^2_{Mp} = A_0 + \frac{1}{\sqrt{n}}A_1 + \frac{1}{n}A_2 + \frac{1}{n\sqrt{n}}A_3 + O_p(n^{-2}),$$

where

$$\begin{aligned}
A_0 =& \mathrm{tr}\boldsymbol{\Lambda} + \boldsymbol{z}^{(1)\prime}_{(34)}\boldsymbol{z}^{(1)}_{(34)}, \\
A_1 =& -\frac{1}{\sqrt{\gamma_1}}(2\boldsymbol{z}^{(1)\prime}_{(12)}\boldsymbol{V}^{(1)}_{(12)(34)}\boldsymbol{z}^{(1)}_{(34)} + \boldsymbol{z}^{(1)\prime}_{(34)}\boldsymbol{V}^{(1)}_{(34)(34)}\boldsymbol{z}^{(1)}_{(34)}) - \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\boldsymbol{\Lambda}\}, \\
A_2 =& \boldsymbol{z}^{(1)\prime}_{(12)}\boldsymbol{z}^{(1)}_{(12)} + \boldsymbol{z}^{(2)\prime}\boldsymbol{z}^{(2)} + \frac{1}{\sqrt{\gamma_1\gamma_2}}\boldsymbol{z}^{(1)\prime}_{(12)}\boldsymbol{z}^{(2)} - \frac{\gamma_2(p_1+p_2)-1}{\gamma_1}\boldsymbol{z}^{(1)\prime}_{(34)}\boldsymbol{z}^{(1)}_{(34)} \\
&+ \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})^2\boldsymbol{\Lambda}\} \\
&- \gamma_1\gamma_2\mathrm{tr}\left\{\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'\boldsymbol{\Lambda}\right\} \\
&+ \frac{1}{\gamma_1}\{\boldsymbol{z}^{(1)\prime}_{(12)}\boldsymbol{V}^{(1)}_{(12)(34)}\boldsymbol{V}^{(1)}_{(34)(12)}\boldsymbol{z}^{(1)}_{(12)} + 2\boldsymbol{z}^{(1)\prime}_{(12)}\boldsymbol{V}^{(1)}_{(12)(12)}\boldsymbol{V}^{(1)}_{(12)(34)}\boldsymbol{z}^{(1)}_{(34)} + 2\boldsymbol{z}^{(1)\prime}_{(12)}\boldsymbol{V}^{(1)}_{(12)(34)}\boldsymbol{V}^{(1)}_{(34)(34)}\boldsymbol{z}^{(1)}_{(34)} \\
&+ \boldsymbol{z}^{(1)\prime}_{(34)}\boldsymbol{V}^{(1)}_{(34)(12)}\boldsymbol{V}^{(1)}_{(12)(34)}\boldsymbol{z}^{(1)}_{(34)} + \boldsymbol{z}^{(1)\prime}_{(34)}(\boldsymbol{V}^{(1)}_{(34)(34)})^2\boldsymbol{z}^{(1)}_{(34)}\},
\end{aligned}$$

$$
\begin{aligned}
A_3 =& \boldsymbol{z}_{(12)}^{(1)\prime}\Big\{ -\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} - \sqrt{\gamma_2}\boldsymbol{V}_{(12)(12)}^{(2)} + \frac{3\gamma_1\gamma_2 - 2}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{V}_{(12)(12)}^{(1)} \\
&+ \frac{\gamma_1\gamma_2 - 1}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)} - \gamma_1(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})^3 \\
&+ 2\gamma_1^2\gamma_2(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)}) \left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_{(12)}^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_{(12)}^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'\Big\}\boldsymbol{z}_{(12)}^{(1)} \\
&+ \boldsymbol{z}_{(12)}^{(1)\prime}\Big\{ -\frac{1}{\sqrt{\gamma_2}}\boldsymbol{V}_{(12)(12)}^{(1)} - \frac{1}{\sqrt{\gamma_1}}\boldsymbol{V}_{(12)(12)}^{(2)} - \frac{2\gamma_1\sqrt{\gamma_2} - \gamma_2\sqrt{\gamma_2}}{\gamma_1}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{V}_{(12)(12)}^{(1)} \\
&- \frac{\gamma_1\sqrt{\gamma_2} - 2\gamma_2\sqrt{\gamma_2}}{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)} - \frac{\gamma_1\sqrt{\gamma_2} - \gamma_2\sqrt{\gamma_2}}{\gamma_1}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)} \\
&+ 2\gamma_1\gamma_2\sqrt{\gamma_1\gamma_2}(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_{(12)}^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_{(12)}^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)' \\
&+ 2\gamma_1\gamma_2\sqrt{\gamma_1\gamma_2}\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_{(12)}^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_{(12)}^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)}) \\
&- 2\sqrt{\gamma_1\gamma_2}(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})^3\Big\}\boldsymbol{z}^{(2)} \\
&+ \boldsymbol{z}^{(2)\prime}\Big\{ -\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} - \sqrt{\gamma_2}\boldsymbol{V}^{(2)} - \frac{3\gamma_2}{\sqrt{\gamma_1}}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{V}_{(12)(12)}^{(1)} \\
&- \frac{\gamma_2}{\sqrt{\gamma_1}}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)} - \gamma_2(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})^3 \\
&+ 2\gamma_1\gamma_2^2(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_{(12)}^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_{(12)}^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'\Big\}\boldsymbol{z}^{(2)} \\
&+ \boldsymbol{z}_{(12)}^{(1)\prime}\Big\{\frac{4\gamma_2^2 - (1-\gamma_1^2)p_{(12)}}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}_{(12)(34)}^{(1)} - \frac{2}{\gamma_1\sqrt{\gamma_1}}(\boldsymbol{V}_{(12)(12)}^{(1)})^2\boldsymbol{V}_{(12)(34)}^{(1)} \\
&- \frac{1+\gamma_1}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)} - \frac{1+\gamma_1}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}_{(12)(34)}^{(1)}(\boldsymbol{V}_{(34)(34)}^{(1)})^2\Big\}\boldsymbol{z}_{(34)}^{(1)} \\
&+ \boldsymbol{z}^{(2)\prime}\Big\{\frac{12\gamma_1^2 - 4\gamma_1 - 1 + \gamma_2^2p_{(12)}}{\gamma_1\sqrt{\gamma_2}}\boldsymbol{V}_{(12)(34)}^{(1)} - \frac{\sqrt{\gamma_2}}{\gamma_1}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)} \\
&- \frac{\sqrt{\gamma_2}}{\gamma_1}\boldsymbol{V}_{(12)(34)}^{(1)}(\boldsymbol{V}_{(34)(34)}^{(1)})^2\Big\}\boldsymbol{z}_{(34)}^{(1)} \\
&+ \boldsymbol{z}_{(34)}^{(1)\prime}\Big\{\frac{\gamma_1 - 2 + \gamma_2p_{(12)}}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}_{(34)(34)}^{(1)} - \frac{1}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{V}_{(12)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)} \\
&- \frac{2}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(34)}^{(1)} - \frac{1}{\gamma_1\sqrt{\gamma_1}}(\boldsymbol{V}_{(34)(34)}^{(1)})^3\Big\}\boldsymbol{z}_{(34)}^{(1)}.
\end{aligned}
$$

Similaly

$$
T_{p_1}^2 = B_0 + \frac{1}{\sqrt{n}}B_1 + \frac{1}{n}B_2 + \frac{1}{n\sqrt{n}}B_3 + O_p(n^{-2}),
$$

where

$$
\begin{aligned}
B_0 =& \mathrm{tr}\boldsymbol{\Lambda}_{11}, \\
B_1 =& -\mathrm{tr}\{(\sqrt{\gamma}_1 \boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}_{11}^{(2)})\boldsymbol{\Lambda}_{11}\}, \\
B_2 =& \gamma_2 \boldsymbol{z}_1^{(1)\prime}\boldsymbol{z}_1^{(1)} + \gamma_1 \boldsymbol{z}_1^{(2)\prime}\boldsymbol{z}_1^{(2)} + \frac{1-2\gamma_1\gamma_2}{\sqrt{\gamma_1\gamma_2}}\boldsymbol{z}_1^{(1)\prime}\boldsymbol{z}_1^{(2)} + \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}_{11}^{(2)})^2\boldsymbol{\Lambda}_{11}\}, \\
B_3 =& \boldsymbol{z}_1^{(1)\prime}\left\{\frac{-3\sqrt{\gamma_1}+2\gamma_1\sqrt{\gamma_1}}{2}\boldsymbol{V}_{11}^{(1)} + \frac{2\gamma_1\gamma_2-\gamma_2-1}{2\sqrt{\gamma_2}}\boldsymbol{V}_{11}^{(2)} - \gamma_1(\sqrt{\gamma_1}\boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}_{11}^{(2)})^3\right\}\boldsymbol{z}_1^{(1)} \\
&+ \boldsymbol{z}_1^{(1)\prime}\left\{\frac{2\gamma_1\gamma_2-\gamma_2-1}{\sqrt{\gamma_2}}\boldsymbol{V}_{11}^{(1)} + \frac{2\gamma_1\gamma_2-\gamma_1-1}{\sqrt{\gamma_1}}\boldsymbol{V}_{11}^{(2)} - 2\sqrt{\gamma_1\gamma_2}(\sqrt{\gamma_1}\boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}_{11}^{(2)})^3\right\}\boldsymbol{z}_1^{(2)} \\
&+ \boldsymbol{z}_1^{(2)\prime}\left\{\frac{2\gamma_1\gamma_2-\gamma_1-1}{2\sqrt{\gamma_1}}\boldsymbol{V}_{11}^{(1)} + \frac{-3\sqrt{\gamma_2}+2\gamma_2\sqrt{\gamma_2}}{2}\boldsymbol{V}_{11}^{(2)} - \gamma_2(\sqrt{\gamma_1}\boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}_{11}^{(2)})^3\right\}\boldsymbol{z}_1^{(2)}.
\end{aligned}
$$

From these results, we can obtain a probability expansion of U_{p_1}.

$$
\begin{aligned}
U_{p_1} =& (N-p)\frac{T_{Mp}^2 - T_{p_1}^2}{N-1+T_{p_1}^2} \\
=& (N-p)\frac{(A_0-B_0) + (A_1-B_1)/\sqrt{n} + (A_2-B_2)/n + (A_3-B_3)/n\sqrt{n} + O_p(n^{-2})}{n + (1+B_0) + B_1/\sqrt{n} + B_2/n + B_3/n\sqrt{n} + O_p(n^{-2})} \\
=& (A_0-B_0) + \frac{1}{\sqrt{n}}(A_1-B_1) + \frac{1}{n}\{(A_2-B_2) + (1-p-B_0)(A_0-B_0)\} \\
&+ \frac{1}{n\sqrt{n}}\{(A_3-B_3) + (2-p)(A_1-B_1) - A_1B_0 - A_0B_1 + 2B_0B_1\} + O_p(n^{-2}) \\
=& Q_0 + \frac{1}{\sqrt{n}}Q_1 + \frac{1}{n}Q_2 + \frac{1}{n\sqrt{n}}Q_3 + O_p(n^{-2}),
\end{aligned}
$$

where

$$
\begin{aligned}
Q_0 =& \mathrm{tr}\boldsymbol{\Lambda}_{22} + \boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{z}_{(34)}^{(1)}, \\
Q_1 =& -\frac{1}{\sqrt{\gamma_1}}(2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + \boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)}) \\
&- \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\boldsymbol{\Lambda}\} + \mathrm{tr}\{(\sqrt{\gamma}_1\boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}_{11}^{(2)})\boldsymbol{\Lambda}_{11}\}, \\
Q_2 =& \mathrm{tr}\boldsymbol{\Lambda}_{11} + \boldsymbol{z}_2^{(1)\prime}\boldsymbol{z}_2^{(1)} + \boldsymbol{z}_2^{(2)\prime}\boldsymbol{z}_2^{(2)} + \frac{1}{\sqrt{\gamma_1\gamma_2}}\boldsymbol{z}_2^{(1)\prime}\boldsymbol{z}_2^{(2)} \\
&+ \left\{\frac{1}{\gamma_1}(1-p_1-p_2) + (1-p_3-p_4)\right\}\boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{z}_{(34)}^{(1)} \\
&- \boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{z}_{(34)}^{(1)}\mathrm{tr}\boldsymbol{\Lambda}_{11} + \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})^2\boldsymbol{\Lambda}\} - \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}_{11}^{(2)})^2\boldsymbol{\Lambda}_{11}\} \\
&- \gamma_1\gamma_2\mathrm{tr}\left\{\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_{(12)}^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_{(12)}^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'\boldsymbol{\Lambda}\right\} + (1-p-\mathrm{tr}\boldsymbol{\Lambda}_{11})\mathrm{tr}\boldsymbol{\Lambda}_{22} \\
&+ \frac{1}{\gamma_1}\{\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{z}_{(12)}^{(1)} + 2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + 2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} \\
&+ \boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + \boldsymbol{z}_{(34)}^{(1)\prime}(\boldsymbol{V}_{(34)(34)}^{(1)})^2\boldsymbol{z}_{(34)}^{(1)}\},
\end{aligned}
$$

$$
\begin{aligned}
Q_3 =& A_3 - B_3 \\
&- \frac{2-p}{\sqrt{\gamma_1}}(2\boldsymbol{z}_{(12)}^{(1)'}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + \boldsymbol{z}_{(34)}^{(1)'}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)}) - (2-p)\mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\boldsymbol{\Lambda}\} \\
&+ (2-p)\mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}_{11}^{(2)})\boldsymbol{\Lambda}_{11}\} \\
&+ \frac{1}{\sqrt{\gamma_1}}(2\boldsymbol{z}_{(12)}^{(1)'}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + \boldsymbol{z}_{(34)}^{(1)'}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)})\mathrm{tr}\boldsymbol{\Lambda}_{11} \\
&+ \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\boldsymbol{\Lambda}\}\mathrm{tr}\boldsymbol{\Lambda}_{11} \\
&+ (\mathrm{tr}\boldsymbol{\Lambda} + \boldsymbol{z}_{(34)}^{(1)'}\boldsymbol{z}_{(34)}^{(1)} - 2\mathrm{tr}\boldsymbol{\Lambda}_{11})\mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}_{11}^{(2)})\boldsymbol{\Lambda}_{11}\}.
\end{aligned}
$$

(A.2.)

We derive the result of the probability expansion of $U_{p_{(123)}}$ as well as U_{p_1}.

$$
T^2_{Mp_{(123)}} = C_0 + \frac{1}{\sqrt{n}}C_1 + \frac{1}{n}C_2 + \frac{1}{n\sqrt{n}}C_3 + O_p(n^{-2}),
$$

where

$$
\begin{aligned}
C_0 =& \mathrm{tr}\boldsymbol{\Lambda} + \boldsymbol{z}_2^{(1)'}\boldsymbol{z}_2^{(1)}, \\
C_1 =& -\frac{1}{\sqrt{\gamma_1}}(2\boldsymbol{z}_1^{(1)'}\boldsymbol{V}_{12}^{(1)}\boldsymbol{z}_2^{(1)} + \boldsymbol{z}_2^{(1)'}\boldsymbol{V}_{22}^{(1)}\boldsymbol{z}_2^{(1)}) - \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\boldsymbol{\Lambda}\}, \\
C_2 =& \boldsymbol{z}_1^{(1)'}\boldsymbol{z}_1^{(1)} + \boldsymbol{z}^{(2)'}\boldsymbol{z}^{(2)} + \frac{1}{\sqrt{\gamma_1\gamma_2}}\boldsymbol{z}_1^{(1)'}\boldsymbol{z}^{(2)} - \frac{\gamma_2 p_1 - 1}{\gamma_1}\boldsymbol{z}_2^{(1)'}\boldsymbol{z}_2^{(1)} \\
&+ \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{11}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})^2\boldsymbol{\Lambda}\} \\
&- \gamma_1\gamma_2\mathrm{tr}\left\{\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_1^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_1^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'\boldsymbol{\Lambda}\right\} \\
&+ \frac{1}{\gamma_1}\{\boldsymbol{z}_1^{(1)'}\boldsymbol{V}_{12}^{(1)}\boldsymbol{V}_{21}^{(1)}\boldsymbol{z}_1^{(1)} + 2\boldsymbol{z}_1^{(1)'}\boldsymbol{V}_{11}^{(1)}\boldsymbol{V}_{12}^{(1)}\boldsymbol{z}_2^{(1)} + 2\boldsymbol{z}_1^{(1)'}\boldsymbol{V}_{12}^{(1)}\boldsymbol{V}_{22}^{(1)}\boldsymbol{z}_2^{(1)} \\
&+ \boldsymbol{z}_2^{(1)'}\boldsymbol{V}_{21}^{(1)}\boldsymbol{V}_{12}^{(1)}\boldsymbol{z}_2^{(1)} + \boldsymbol{z}_2^{(1)'}(\boldsymbol{V}_{22}^{(1)})^2\boldsymbol{z}_2^{(1)}\},
\end{aligned}
$$

$$
\begin{aligned}
C_3 =& \boldsymbol{z}^{(1)\prime}_{(12)}\Big\{ -\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} - \sqrt{\gamma_2}\boldsymbol{V}^{(2)}_{(12)(12)} + \frac{3\gamma_1\gamma_2-2}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}^{(1)}_{(12)3}\boldsymbol{V}^{(1)}_{3(12)}\boldsymbol{V}^{(1)}_{(12)(12)} \\
&+ \frac{\gamma_1\gamma_2-1}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}^{(1)}_{(12)3}\boldsymbol{V}^{(1)}_{33}\boldsymbol{V}^{(1)}_{3(12)} - \gamma_1(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})^3 \\
&+ 2\gamma_1^2\gamma_2(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'\Big\}\boldsymbol{z}^{(1)}_{(12)} \\
&+ \boldsymbol{z}^{(1)\prime}_{(12)}\Big\{ -\frac{1}{\sqrt{\gamma_2}}\boldsymbol{V}^{(1)}_{(12)(12)} - \frac{1}{\sqrt{\gamma_1}}\boldsymbol{V}^{(2)}_{(12)(12)} - \frac{2\gamma_1\sqrt{\gamma_2}-\gamma_2\sqrt{\gamma_2}}{\gamma_1}\boldsymbol{V}^{(1)}_{(12)3}\boldsymbol{V}^{(1)}_{3(12)}\boldsymbol{V}^{(1)}_{(12)(12)} \\
&- \frac{\gamma_1\sqrt{\gamma_2}-2\gamma_2\sqrt{\gamma_2}}{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)}\boldsymbol{V}^{(1)}_{(12)3}\boldsymbol{V}^{(1)}_{3(12)} - \frac{\gamma_1\sqrt{\gamma_2}-\gamma_2\sqrt{\gamma_2}}{\gamma_1}\boldsymbol{V}^{(1)}_{(12)3}\boldsymbol{V}^{(1)}_{33}\boldsymbol{V}^{(1)}_{3(12)} \\
&+ 2\gamma_1\gamma_2\sqrt{\gamma_1\gamma_2}(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)' \\
&+ 2\gamma_1\gamma_2\sqrt{\gamma_1\gamma_2}\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)}) \\
&- 2\sqrt{\gamma_1\gamma_2}(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})^3\Big\}\boldsymbol{z}^{(2)} \\
&+ \boldsymbol{z}^{(2)\prime}\Big\{ -\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} - \sqrt{\gamma_2}\boldsymbol{V}^{(2)} - \frac{3\gamma_2}{\sqrt{\gamma_1}}\boldsymbol{V}^{(1)}_{(12)3}\boldsymbol{V}^{(1)}_{3(12)}\boldsymbol{V}^{(1)}_{(12)(12)} \\
&- \frac{\gamma_2}{\sqrt{\gamma_1}}\boldsymbol{V}^{(1)}_{(12)3}\boldsymbol{V}^{(1)}_{33}\boldsymbol{V}^{(1)}_{3(12)} - \gamma_2(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})^3 \\
&+ 2\gamma_1\gamma_2^2(\sqrt{\gamma_1}\boldsymbol{V}^{(1)}_{(12)(12)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}^{(1)}_{(12)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'\Big\}\boldsymbol{z}^{(2)} \\
&+ \boldsymbol{z}^{(1)\prime}_{(12)}\Big\{\frac{4\gamma_2^2-(1-\gamma_1^2)p_{(12)}}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}^{(1)}_{(12)3} - \frac{2}{\gamma_1\sqrt{\gamma_1}}(\boldsymbol{V}^{(1)}_{(12)(12)})^2\boldsymbol{V}^{(1)}_{(12)3} \\
&- \frac{1+\gamma_1}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}^{(1)}_{(12)3}\boldsymbol{V}^{(1)}_{3(12)}\boldsymbol{V}^{(1)}_{(12)3} - \frac{1+\gamma_1}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}^{(1)}_{(12)3}(\boldsymbol{V}^{(1)}_{33})^2\Big\}\boldsymbol{z}^{(1)}_3 \\
&+ \boldsymbol{z}^{(2)\prime}\Big\{\frac{12\gamma_1^2-4\gamma_1-1+\gamma_2^2p_{(12)}}{\gamma_1\sqrt{\gamma_2}}\boldsymbol{V}^{(1)}_{(12)3} - \frac{\sqrt{\gamma_2}}{\gamma_1}\boldsymbol{V}^{(1)}_{(12)3}\boldsymbol{V}^{(1)}_{3(12)}\boldsymbol{V}^{(1)}_{(12)3} - \frac{\sqrt{\gamma_2}}{\gamma_1}\boldsymbol{V}^{(1)}_{(12)3}(\boldsymbol{V}^{(1)}_{33})^2\Big\}\boldsymbol{z}^{(1)}_3 \\
&+ \boldsymbol{z}^{(1)\prime}_3\Big\{\frac{\gamma_1-2+\gamma_2p_{(12)}}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}^{(1)}_{33} - \frac{1}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}^{(1)}_{3(12)}\boldsymbol{V}^{(1)}_{(12)(12)}\boldsymbol{V}^{(1)}_{(12)3} \\
&- \frac{2}{\gamma_1\sqrt{\gamma_1}}\boldsymbol{V}^{(1)}_{3(12)}\boldsymbol{V}^{(1)}_{(12)3}\boldsymbol{V}^{(1)}_{33} - \frac{1}{\gamma_1\sqrt{\gamma_1}}(\boldsymbol{V}^{(1)}_{33})^3\Big\}\boldsymbol{z}^{(1)}_3.
\end{aligned}
$$

From this results, we can obtain a probability expansion of $U_{p_{(123)}}$.

$$
\begin{aligned}
U_{p_{(123)}} =&(N-p)\frac{T^2_{Mp}-T^2_{p_{(123)}}}{N-1+T^2_{p_{(123)}}} \\
=&(A_0-C_0)+\frac{1}{\sqrt{n}}(A_1-C_1)+\frac{1}{n}\{(A_2-C_2)+(1-p-C_0)(A_0-C_0)\} \\
&+\frac{1}{n\sqrt{n}}\{(A_3-C_3)+(2-p)(A_1-C_1)-A_1C_0-A_0C_1+2C_0C_1\}+O_p(n^{-2}) \\
=&R_0+\frac{1}{\sqrt{n}}R_1+\frac{1}{n}R_2+\frac{1}{n\sqrt{n}}R_3+O_p(n^{-2}),
\end{aligned}
$$

where

$$
\begin{aligned}
R_0 =& \boldsymbol{z}_4^{(1)\prime}\boldsymbol{z}_4^{(1)},\\
R_1 =& -\frac{1}{\sqrt{\gamma_1}}(2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)4}^{(1)}\boldsymbol{z}_4^{(1)} + \boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} - \boldsymbol{z}_3^{(1)\prime}\boldsymbol{V}_{33}^{(1)}\boldsymbol{z}_3^{(1)}),\\
R_2 =& \left\{(1-p) - \frac{1}{\gamma_1}(\gamma_2 p_{(12)} + 1)\right\}\boldsymbol{z}_4^{(1)\prime}\boldsymbol{z}_4^{(1)} - \boldsymbol{z}_4^{(1)\prime}\boldsymbol{z}_4^{(1)}(\mathrm{tr}\boldsymbol{\Lambda}_{(12)(12)} + \boldsymbol{z}_3^{(1)\prime}\boldsymbol{z}_3^{(1)})\\
&+ \frac{1}{\gamma_1}\{\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{z}_{(12)}^{(1)} + 2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)}\\
&+ 2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + \boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{V}_{(34)(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + \boldsymbol{z}_{(34)}^{(1)\prime}(\boldsymbol{V}_{(34)(34)}^{(1)})^2\boldsymbol{z}_{(34)}^{(1)}\}\\
&- \frac{1}{\gamma_1}\{\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)3}^{(1)}\boldsymbol{V}_{3(12)}^{(1)}\boldsymbol{z}_{(12)}^{(1)} + 2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)(12)}^{(1)}\boldsymbol{V}_{(12)3}^{(1)}\boldsymbol{z}_3^{(1)}\\
&+ 2\boldsymbol{z}_{(12)}^{(1)\prime}\boldsymbol{V}_{(12)3}^{(1)}\boldsymbol{V}_{33}^{(1)}\boldsymbol{z}_3^{(1)} + \boldsymbol{z}_3^{(1)\prime}\boldsymbol{V}_{3(12)}^{(1)}\boldsymbol{V}_{(12)3}^{(1)}\boldsymbol{z}_3^{(1)} + \boldsymbol{z}_3^{(1)\prime}(\boldsymbol{V}_{33}^{(1)})^2\boldsymbol{z}_3^{(1)}\},\\
R_3 =& A_3 - C_3\\
&- \frac{2-p}{\sqrt{\gamma_1}}(2\boldsymbol{z}_{(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + \boldsymbol{z}_{(34)}^{(1)}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)}) + \frac{2-p}{\sqrt{\gamma_1}}(2\boldsymbol{z}_{(12)}^{(1)}\boldsymbol{V}_{(12)3}^{(1)}\boldsymbol{z}_3^{(1)} + \boldsymbol{z}_3^{(1)}\boldsymbol{V}_{33}^{(1)}\boldsymbol{z}_3^{(1)})\\
&+ \frac{1}{\sqrt{\gamma_1}}(2\boldsymbol{z}_{(12)}^{(1)}\boldsymbol{V}_{(12)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)} + \boldsymbol{z}_{(34)}^{(1)}\boldsymbol{V}_{(34)(34)}^{(1)}\boldsymbol{z}_{(34)}^{(1)})(\mathrm{tr}\boldsymbol{\Lambda} + \boldsymbol{z}_3^{(1)\prime}\boldsymbol{z}_3^{(1)\prime})\\
&+ \frac{1}{\sqrt{\gamma_1}}(2\boldsymbol{z}_{(12)}^{(1)}\boldsymbol{V}_{(12)3}^{(1)}\boldsymbol{z}_3^{(1)} + \boldsymbol{z}_3^{(1)}\boldsymbol{V}_{33}^{(1)}\boldsymbol{z}_3^{(1)})(2\mathrm{tr}\boldsymbol{\Lambda} + \boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{z}_{(34)}^{(1)\prime} + \boldsymbol{z}_3^{(1)\prime}\boldsymbol{z}_3^{(1)\prime})\\
&+ \mathrm{tr}\{(\sqrt{\gamma_1}\boldsymbol{V}_{(12)(12)}^{(1)} + \sqrt{\gamma_2}\boldsymbol{V}^{(2)})\boldsymbol{\Lambda}\}(3\mathrm{tr}\boldsymbol{\Lambda} + \boldsymbol{z}_{(34)}^{(1)\prime}\boldsymbol{z}_{(34)}^{(1)\prime} + 2\boldsymbol{z}_3^{(1)\prime}\boldsymbol{z}_3^{(1)\prime}).
\end{aligned}
$$

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