Tests for Profile Analysis in Behrens-Fisher Problem

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Abstract

In this study, we consider profile analysis with unequal covariance matrices. We propose test statistics for testing parallelism, level and flatness hypotheses in a two-sample problem with unequal covariance matrices. We derive upper percentiles of these test statistics. Moreover, we propose modified Bartlett corrected statistics for three hypotheses. The approximate accuracy is evaluated via Monte Carlo simulations for some selected parameter values. Finally, a numerical example is presented to illustrate our proposed methods.

Keywords: Asymptotic expansion; Modified Bartlett correction; Monte Carlo simulation; Unequal covariance matrices; Profile analysis.

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1 Introduction

Profile analysis is a statistical method for comparing the profiles of two or more groups. It includes three types of hypothesis tests: parallelism, level, and flatness. In recent years, profile analysis has been studied by many authors. Profile analysis under multivariate normality with equal covariance matrices has been introduced by Morrison (2002) and others. In the non-normal case, Okamoto et al. (2006) discussed profile analysis under elliptical distributions, and Maruyama (2007) obtained asymptotic expansions for the null distributions of some test statistics under general distributions. Profile analysis for two-step monotone missing data in a multivariate normal population was discussed by Onozawa et al. (2013). Recently, Cengiz et al. (2021) considered profile analysis from a high-dimensional perspective and derived the exact null distributions for the three hypotheses.

The problem of testing hypotheses about two mean vectors under unequal covariance matrices is commonly referred to as the multivariate Behrens–Fisher problem. Under the assumption of multivariate normality, Yanagihara and Yuan (2005) proposed three approximate solutions to the multivariate Behrens-Fisher problem: the F-approximation, the Bartlett correction, and the modified Bartlett correction. Kawasaki and Seo (2015) improved these methods by adjusting the degrees of freedom in the F-distribution and using bias correction with higher-order asymptotic expansions. The multivariate Behrens–Fisher problem under non-normality has also been discussed by Kakizawa and Iwashita (2008), who derived asymptotic expansions under local alternatives and general distributions. They presented theoretical results without numerical evaluation of approximation accuracy. Notably, the test statistic used for the parallelism hypothesis in profile analysis is essentially the same as that used for testing the equality of mean vectors.

Throughout this paper, we consider profile analysis under multivariate normality with unequal covariance matrices. We propose test statistics for the hypotheses of parallelism, level, and flatness. Since it is difficult to give an exact distribution, an asymptotic expansion of the null distribution for these statistics is derived via the perturbation method. We obtain approximate upper percentiles for the distribution of test statistics.

The organization of the paper is as follows. In Section 2, profile analysis with unequal covariance matrices is considered. Monte Carlo simulation is performed to investigate the accuracy of the approximation to the null distribution of these statistics in Section 3. Section 4 provides a numerical example to illustrate the proposed methods. Finally, in Section 5, the conclusion is presented. Technical details are given in the Appendix.

2 Profile analysis with unequal covariance matrices

In this section, we consider profile analysis under unequal covariance matrices. Let $\boldsymbol{x}_1^{(k)}, \ldots, \boldsymbol{x}_{N_k}^{(k)}$ be p-dimensional random vectors from $N_p(\boldsymbol{\mu}^{(k)}, \Sigma^{(k)}), k = 1, 2$, where $\boldsymbol{\mu}^{(k)} = (\mu_1^{(k)}, \ldots, \mu_p^{(k)})'$ and $\Sigma^{(1)} \neq \Sigma^{(2)}$. The sample mean vector and the sample covariance matrix are defined by

$$\overline{\boldsymbol{x}}^{(k)} = \frac{1}{N_k}\sum_{j=1}^{N_k} \boldsymbol{x}_j^{(k)}, \quad S^{(k)} = \frac{1}{N_k - 1}\sum_{j=1}^{N_k}(\boldsymbol{x}_j^{(k)} - \overline{\boldsymbol{x}}^{(k)})(\boldsymbol{x}_j^{(k)} - \overline{\boldsymbol{x}}^{(k)})',$$

respectively. By the assumption of the standard regularity condition $N_k/N = O(1), k = 1, 2$, we derive upper percentiles for the distribution of test statistics.

2.1 Parallelism hypothesis

We first consider the parallelism hypothesis expressed as

$$H_{10} : C\boldsymbol{\mu}^{(1)} = C\boldsymbol{\mu}^{(2)} \text{ vs. } H_{11} : C\boldsymbol{\mu}^{(1)} \neq C\boldsymbol{\mu}^{(2)},$$

where C is a $(p-1) \times p$ matrix of rank $p-1$ such that $C\boldsymbol{1}_p = \boldsymbol{0}$ and $\boldsymbol{1}_p$ is a p-vector of ones. The test statistic under the null hypothesis H_{10} is proposed as follows:

$$T_1^2 = (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})'C'\left(\frac{1}{N_1}CS^{(1)}C' + \frac{1}{N_2}CS^{(2)}C'\right)^{-1} C(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)}).$$

Let

$$\overline{\Sigma}_1 = \frac{1}{N_1}\Sigma^{(1)} + \frac{1}{N_2}\Sigma^{(2)}, \quad \boldsymbol{z}_1 = (C\overline{\Sigma}_1C')^{-\frac{1}{2}}C(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)}),$$

$$W_1 = (C\overline{\Sigma}_1C')^{-\frac{1}{2}}\left(\frac{1}{N_1}CS^{(1)}C' + \frac{1}{N_2}CS^{(2)}C'\right)(C\overline{\Sigma}_1C')^{-\frac{1}{2}}.$$

Then T_1^2 can be rewritten as

$$T_1^2 = \boldsymbol{z}_1' W_1^{-1} \boldsymbol{z}_1.$$

By the perturbation method, T_1^2 can be expressed as

$$T_1^2 = \boldsymbol{z}_1'\boldsymbol{z}_1 - \frac{1}{\sqrt{n}}\boldsymbol{z}_1' V_1 \boldsymbol{z}_1 + \frac{1}{n}\boldsymbol{z}_1' V_1^2 \boldsymbol{z}_1 + O_p(n^{-\frac{3}{2}}),$$

where $n = N - 2$. The detailed derivation of T_1^2 and the definition of V_1 are provided in Appendix A.1. The characteristic function of T_1^2 can be expanded as follows:

$$E[\exp(itT_1^2)] = u^{-\frac{p-1}{2}} + \frac{1}{n}\sum_{j=0}^{2} \beta_{1j} u^{-\frac{p-1}{2}-j} + O(n^{-\frac{3}{2}}),$$

where

$$\begin{aligned}
&u = 1 - 2it, \quad \beta_{10} = -\frac{1}{4}a_1, \quad \beta_{11} = -\frac{1}{2}b_1, \quad \beta_{12} = \frac{1}{4}(a_1 + 2b_1),\\
&a_1 = \rho_1^{-2}(\mathrm{tr}\Omega_{11}\Omega_{11}')^2 + \rho_2^{-2}(\mathrm{tr}\Omega_{12}\Omega_{12}')^2, \quad b_1 = \rho_1^{-2}\mathrm{tr}(\Omega_{11}\Omega_{11}')^2 + \rho_2^{-2}\mathrm{tr}(\Omega_{12}\Omega_{12}')^2.
\end{aligned}$$

The definitions of Ω_{11} and Ω_{12} are presented in Appendix A.1. Appendix A.2 provides the derivation of $E[\exp(itT_1^2)]$ and the definitions of ρ_1 and ρ_2. The distribution of T_1^2 is given by

$$\Pr(T_1^2 \le x) = G_{p-1}(x) + \frac{1}{n}\sum_{j=0}^{2}\beta_{1j} G_{p-1+2j}(x) + O(n^{-\frac{3}{2}}),$$

where $G_f(x)$ is the distribution function of the χ^2 distribution with f degrees of freedom. The upper 100α percentile of T_1^2 can be expanded as

$$t_1^2(\alpha) = \chi^2_{p-1}(\alpha) - \frac{2\chi^2_{p-1}(\alpha)}{n(p-1)}\left(\beta_{10} - \frac{\beta_{12}}{p+1}\chi^2_{p-1}(\alpha)\right) + O(n^{-\frac{3}{2}}),$$

where $\chi^2_f(\alpha)$ is the upper 100α percentile of the χ^2 distribution with f degrees of freedom. The expression is approximated by replacing β_{10} and β_{12} with their respective estimated values.

$$\widehat{t_1^2}(\alpha) = \chi^2_{p-1}(\alpha) - \frac{2\chi^2_{p-1}(\alpha)}{n(p-1)}\left(\widehat{\beta}_{10} - \frac{\widehat{\beta}_{12}}{p+1}\chi^2_{p-1}(\alpha)\right),$$

where

$$\begin{aligned}
&\widehat{\beta}_{10} = -\frac{1}{4}\widehat{a}_1, \quad \widehat{\beta}_{11} = -\frac{1}{2}\widehat{b}_1, \quad \widehat{\beta}_{12} = \frac{1}{4}(\widehat{a}_1 + 2\widehat{b}_1),\\
&\widehat{a}_1 = \rho_1^{-2}(\mathrm{tr}\widehat{\Omega}_{11}\widehat{\Omega}_{11}')^2 + \rho_2^{-2}(\mathrm{tr}\widehat{\Omega}_{12}\widehat{\Omega}_{12}')^2, \quad \widehat{b}_1 = \rho_1^{-2}\mathrm{tr}(\widehat{\Omega}_{11}\widehat{\Omega}_{11}')^2 + \rho_2^{-2}\mathrm{tr}(\widehat{\Omega}_{12}\widehat{\Omega}_{12}')^2,\\
&\widehat{\Omega}_{1i} = \frac{1}{\sqrt{N_i}}\left\{C\left(\frac{1}{N_1}S^{(1)} + \frac{1}{N_2}S^{(2)}\right)C'\right\}^{-\frac{1}{2}}(CS^{(i)}C')^{\frac{1}{2}}.
\end{aligned}$$

To improve the χ^2 approximation of the null distribution for T_1^2, we apply the modified Bartlett correction proposed by Fujikoshi (2000). Whereas the Bartlett correction matches the first moment of T_1^2 to that of the limiting χ^2 distribution, the modified Bartlett correction aligns both the first and second moments and eliminates the $O(n^{-1})$ term in the expansion of the characteristic function of the null distribution, thereby achieving second-order accuracy in distribution. Accordingly, the modified Bartlett correction is adopted throughout this study.

For the parallelism hypothesis, we give the following modified Bartlett–corrected statistic:

$$T_{1B}^2 = (n\gamma_{11} + \gamma_{12}) \log \left(1 + \frac{T_1^2}{n\gamma_{11}}\right),$$

where

$$\gamma_{11} = \frac{2}{c_{12} - 2c_{11}}, \quad \gamma_{12} = \frac{(p+1)c_{12} - 2(p+3)c_{11}}{2(c_{12} - 2c_{11})},$$
$$c_{11} = \frac{1}{p-1}\sum_{j=0}^{2} \widehat{\beta}_{1j}(p-1+2j), \quad c_{12} = \frac{1}{(p-1)(p+1)}\sum_{j=0}^{2} \widehat{\beta}_{1j}(p-1+2j)(p+1+2j).$$

T_{1B}^2 is asymptotically distributed as a chi-squared distribution χ_{p-1}^2 with $p-1$ degrees of freedom.

2.2 Level hypothesis

If the parallelism hypothesis holds, we test the level hypothesis or flatness hypothesis. The level hypothesis is expressed as

$$H_{20}|H_{10} : \mathbf{1}_p'\boldsymbol{\mu}^{(1)} = \mathbf{1}_p'\boldsymbol{\mu}^{(2)} \text{ vs. } H_{21}|H_{10} : \mathbf{1}_p'\boldsymbol{\mu}^{(1)} \neq \mathbf{1}_p'\boldsymbol{\mu}^{(2)},$$

where $H_{20}|H_{10}$ means H_{20} under the assumption that H_{10} is true. The test statistic under the $H_{20}|H_{10}$ is proposed as follows:

$$T_2^2 = \mathbf{1}_p'(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)}) \left(\frac{1}{N_1}\mathbf{1}_p'S^{(1)}\mathbf{1}_p + \frac{1}{N_2}\mathbf{1}_p'S^{(2)}\mathbf{1}_p\right)^{-1} (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})'\mathbf{1}_p.$$

Let

$$z_2 = (\mathbf{1}_p'\overline{\Sigma}_1\mathbf{1}_p)^{-\frac{1}{2}}\mathbf{1}_p'(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)}),$$
$$w_2 = (\mathbf{1}_p'\overline{\Sigma}_1\mathbf{1}_p)^{-\frac{1}{2}} \left(\frac{1}{N_1}\mathbf{1}_p'S^{(1)}\mathbf{1}_p + \frac{1}{N_2}\mathbf{1}_p'S^{(2)}\mathbf{1}_p\right) (\mathbf{1}_p'\overline{\Sigma}_1\mathbf{1}_p)^{-\frac{1}{2}}.$$

Then we can rewrite the latter equation as

$$T_2^2 = \frac{z_2^2}{w_2}.$$

By the perturbation method, T_2^2 can be written as

$$T_2^2 = z_2^2 - \frac{1}{\sqrt{n}}z_2^2V_2 + \frac{1}{n}z_2^2V_2^2 + O_p(n^{-\frac{3}{2}}),$$

where

$$V_2 = \sum_{i=1}^{2} \rho_i^{-1} \Omega_{2i}^2 V_2^{(i)}, \quad \Omega_{2i} = \frac{1}{\sqrt{N_i}} (\mathbf{1}_p' \overline{\Sigma}_1 \mathbf{1}_p)^{-\frac{1}{2}} (\mathbf{1}_p' \Sigma^{(i)} \mathbf{1}_p)^{\frac{1}{2}},$$

$$V_2^{(i)} = \sqrt{N_i - 1} \left[(\mathbf{1}_p' \Sigma^{(i)} \mathbf{1}_p)^{-1} (\mathbf{1}_p' S^{(i)} \mathbf{1}_p) - 1 \right].$$

The above expansion can be derived in a similar manner to T_1^2, detailed in Appendix A.1. The characteristic function of T_2^2 can be expanded as follows:

$$E[\exp(itT_2^2)] = u^{-\frac{1}{2}} + \frac{1}{n} \sum_{j=0}^{2} \beta_{2j} u^{-\frac{1}{2}-j} + O(n^{-\frac{3}{2}}),$$

where

$$\beta_{20} = -\frac{1}{4} a_2, \quad \beta_{21} = -\frac{1}{2} a_2, \quad \beta_{22} = \frac{3}{4} a_2, \quad a_2 = \rho_1^{-2} \Omega_{21}^4 + \rho_2^{-2} \Omega_{22}^4.$$

The expansion of the characteristic function is obtained in a similar manner as in the case of T_1^2, as shown in Appendix A.2. The distribution of T_2^2 is given by

$$\Pr(T_2^2 \leq x) = G_1(x) + \frac{1}{n} \sum_{j=0}^{2} \beta_{2j} G_{1+2j}(x) + O(n^{-\frac{3}{2}}),$$

where $G_f(x)$ is the distribution function of the χ^2 distribution with f degrees of freedom. The upper 100α percentile of T_2^2 can be expanded as

$$t_2^2(\alpha) = \chi_1^2(\alpha) - \frac{2}{n} \chi_1^2(\alpha) \left(\beta_{20} - \frac{\beta_{22}}{3} \chi_1^2(\alpha) \right) + O(n^{-\frac{3}{2}}).$$

The expression is approximated by substituting β_{20} and β_{22} with their respective estimates.

$$\widehat{t_2^2}(\alpha) = \chi_1^2(\alpha) - \frac{2}{n} \chi_1^2(\alpha) \left(\widehat{\beta}_{20} - \frac{\widehat{\beta}_{22}}{3} \chi_1^2(\alpha) \right),$$

where

$$\widehat{\beta}_{20} = -\frac{1}{4} \widehat{a}_2, \quad \widehat{\beta}_{21} = -\frac{1}{2} \widehat{a}_2, \quad \widehat{\beta}_{22} = \frac{3}{4} \widehat{a}_2,$$

$$\widehat{a}_2 = \rho_1^{-2} \widehat{\Omega}_{21}^4 + \rho_2^{-2} \widehat{\Omega}_{22}^4,$$

$$\widehat{\Omega}_{2i} = \frac{1}{\sqrt{N_i}} \left\{ \mathbf{1}_p' \left(\frac{1}{N_1} S^{(1)} + \frac{1}{N_2} S^{(2)} \right) \mathbf{1}_p \right\}^{-\frac{1}{2}} (\mathbf{1}_p' S^{(i)} \mathbf{1}_p)^{\frac{1}{2}}.$$

For the level hypothesis, we give the following modified Bartlett–corrected statistic:

$$T_{2B}^2 = (n\gamma_{21} + \gamma_{22}) \log \left(1 + \frac{T_2^2}{n\gamma_{21}} \right),$$

where

$$\gamma_{21} = \frac{2}{c_{22} - 2c_{21}}, \quad \gamma_{22} = \frac{3c_{22} - 10c_{21}}{2(c_{22} - 2c_{21})},$$

$$c_{21} = \sum_{j=0}^{2} \widehat{\beta}_{2j}(1 + 2j), \quad c_{22} = \frac{1}{3} \sum_{j=0}^{2} \widehat{\beta}_{2j}(1 + 2j)(3 + 2j).$$

T_{2B}^2 is asymptotically distributed as a chi-squared distribution χ_1^2.

2.3 Flatness hypothesis

The flatness hypothesis is expressed as

$$H_{30}|H_{10} : C(\boldsymbol{\mu}^{(1)} + \boldsymbol{\mu}^{(2)}) = \mathbf{0} \,\text{vs.}\, H_{31}|H_{10} : C(\boldsymbol{\mu}^{(1)} + \boldsymbol{\mu}^{(2)}) \neq \mathbf{0}.$$

The test statistic under the $H_{30}|H_{10}$ is proposed as

$$T_3^2 = N\overline{\boldsymbol{x}}_{12}'C'\left(\frac{N_1}{N}CS^{(1)}C' + \frac{N_2}{N}CS^{(2)}C'\right)^{-1}C\overline{\boldsymbol{x}}_{12}.$$

Let

$$\overline{\Sigma}_2 = \frac{N_1}{N}\Sigma^{(1)} + \frac{N_2}{N}\Sigma^{(2)}, \quad \boldsymbol{z}_3 = \sqrt{N}(C\overline{\Sigma}_2C')^{-\frac{1}{2}}C\overline{\boldsymbol{x}}_{12},$$
$$W_3 = (C\overline{\Sigma}_2C')^{-\frac{1}{2}}\left(\frac{N_1}{N}CS^{(1)}C' + \frac{N_2}{N}CS^{(2)}C'\right)(C\overline{\Sigma}_2C')^{-\frac{1}{2}}.$$

Then we can rewrite the statistic as follows:

$$T_3^2 = \boldsymbol{z}_3'W_3^{-1}\boldsymbol{z}_3.$$

Using the perturbation method, T_3^2 can be expressed as

$$T_3^2 = \boldsymbol{z}_3'\boldsymbol{z}_3 - \frac{1}{\sqrt{n}}\boldsymbol{z}_3'V_3\boldsymbol{z}_3 + \frac{1}{n}\boldsymbol{z}_3'V_3^2\boldsymbol{z}_3 + O_p(n^{-\frac{3}{2}}),$$

where

$$V_3 = \sum_{i=1}^{2}\rho_i^{-1}\Omega_{3i}V_3^{(i)}\Omega_{3i}', \quad \Omega_{3i} = \sqrt{\frac{N_i}{N}}(C\overline{\Sigma}_2C')^{-\frac{1}{2}}(C\Sigma^{(i)}C')^{\frac{1}{2}},$$
$$V_3^{(i)} = \sqrt{N_i - 1}\left[(C\Sigma^{(i)}C')^{-\frac{1}{2}}CS^{(i)}C'(C\Sigma^{(i)}C')^{-\frac{1}{2}} - I_{p-1}\right].$$

The expansion can be derived in a similar way to the derivation of T_1^2, as detailed in Appendix A.1. The characteristic function of T_3^2 can be expanded as follows:

$$E[\exp(itT_3^2)] = u^{-\frac{p-1}{2}} + \frac{1}{n}\sum_{j=0}^{2}\beta_{3j}u^{-\frac{p-1}{2}-j} + O(n^{-\frac{3}{2}}),$$

where

$$\beta_{30} = -\frac{1}{4}a_3, \quad \beta_{31} = -\frac{1}{2}b_3, \quad \beta_{32} = \frac{1}{4}(a_3 + 2b_3),$$
$$a_3 = \rho_1^{-2}(\text{tr}\Omega_{31}\Omega_{31}')^2 + \rho_2^{-2}(\text{tr}\Omega_{32}\Omega_{32}')^2,$$
$$b_3 = \rho_1^{-2}\text{tr}(\Omega_{31}\Omega_{31}')^2 + \rho_2^{-2}\text{tr}(\Omega_{32}\Omega_{32}')^2.$$

The expansion of the characteristic function is obtained in a similar manner to that of T_1^2, as shown in Appendix A.2. The distribution of T_3^2 is given by

$$\Pr(T_3^2 \leq x) = G_{p-1}(x) + \frac{1}{n}\sum_{j=0}^{2}\beta_{3j}G_{p-1+2j}(x) + O(n^{-\frac{3}{2}}),$$

where $G_f(x)$ is the distribution function of the χ^2 distribution with f degrees of freedom. The upper 100α percentile of T_3^2 can be expanded as

$$t_3^2(\alpha) = \chi_{p-1}^2(\alpha) - \frac{2\chi_{p-1}^2(\alpha)}{n(p-1)}\left(\beta_{30} - \frac{\beta_{32}}{p+1}\chi_{p-1}^2(\alpha)\right) + O(n^{-\frac{3}{2}}).$$

By substituting β_{30} and β_{32} with their estimates $\widehat{\beta}_{30}$ and $\widehat{\beta}_{32}$, the expression can be approximated as follows.

$$\widehat{t_3^2}(\alpha) = \chi_{p-1}^2(\alpha) - \frac{2\chi_{p-1}^2(\alpha)}{n(p-1)}\left(\widehat{\beta}_{30} - \frac{\widehat{\beta}_{32}}{p+1}\chi_{p-1}^2(\alpha)\right),$$

where

$$\widehat{\beta}_{30} = -\frac{1}{4}\widehat{a}_3, \quad \widehat{\beta}_{31} = -\frac{1}{2}\widehat{b}_3, \quad \widehat{\beta}_{32} = \frac{1}{4}(\widehat{a}_3 + 2\widehat{b}_3),$$
$$\widehat{a}_3 = \rho_1^{-2}(\mathrm{tr}\widehat{\Omega}_{31}\widehat{\Omega}_{31}')^2 + \rho_2^{-2}(\mathrm{tr}\widehat{\Omega}_{32}\widehat{\Omega}_{32}')^2, \quad \widehat{b}_3 = \rho_1^{-2}\mathrm{tr}(\widehat{\Omega}_{31}\widehat{\Omega}_{31}')^2 + \rho_2^{-2}\mathrm{tr}(\widehat{\Omega}_{32}\widehat{\Omega}_{32}')^2,$$
$$\widehat{\Omega}_{3i} = \sqrt{\frac{N_i}{N}}\left\{C\left(\frac{N_1}{N}S^{(1)} + \frac{N_2}{N}S^{(2)}\right)C'\right\}^{-\frac{1}{2}}(CS^{(i)}C')^{\frac{1}{2}}.$$

For the flatness hypothesis, we give the following modified Bartlett–corrected statistic:

$$T_{3B}^2 = (n\gamma_{31} + \gamma_{32})\log\left(1 + \frac{T_3^2}{n\gamma_{31}}\right),$$

where

$$\gamma_{31} = \frac{2}{c_{32} - 2c_{31}}, \quad \gamma_{32} = \frac{(p+1)c_{32} - 2(p+3)c_{31}}{2(c_{32} - 2c_{31})},$$
$$c_{31} = \frac{1}{p-1}\sum_{j=0}^{2}\widehat{\beta}_{3j}(p-1+2j), \quad c_{32} = \frac{1}{(p-1)(p+1)}\sum_{j=0}^{2}\widehat{\beta}_{3j}(p-1+2j)(p+1+2j).$$

T_{3B}^2 is asymptotically distributed as a chi-squared distribution χ_{p-1}^2.

3 Simulation studies

In this section, we perform a Monte Carlo simulation to investigate the accuracy of the upper 100α percentile of $T_1^2, T_2^2, T_3^2, T_{1B}^2, T_{2B}^2, T_{3B}^2$ statistics for selected parameter values. In the numerical experiment, we carry out 1,000,000 replications based on normal random vectors and choose $\alpha = 0.05, 0.01, p = 3, 8$. We compare the following type I errors:

$$\alpha_1 = \Pr\big(T_1^2 > \chi_{p-1}^2(\alpha)\big), \quad \alpha_2 = \Pr\big(T_1^2 > \widehat{t_1^2}(\alpha)\big), \quad \alpha_3 = \Pr\big(T_{1B}^2 > \chi_{p-1}^2(\alpha)\big),$$
$$\alpha_4 = \Pr\big(T_2^2 > \chi_1^2(\alpha)\big), \quad \alpha_5 = \Pr\big(T_2^2 > \widehat{t_2^2}(\alpha)\big), \quad \alpha_6 = \Pr\big(T_{2B}^2 > \chi_1^2(\alpha)\big),$$
$$\alpha_7 = \Pr\big(T_3^2 > \chi_{p-1}^2(\alpha)\big), \quad \alpha_8 = \Pr\big(T_3^2 > \widehat{t_3^2}(\alpha)\big), \quad \alpha_9 = \Pr\big(T_{3B}^2 > \chi_{p-1}^2(\alpha)\big)$$

Tables 1–4 present the empirical size $\widehat{\alpha}_{ij}, i = 1,2,3, j = 1,2,3$ in the case of $\Sigma^{(1)} = \sigma^2 I$ and $\Sigma^{(2)} = I$, where $\sigma^2 = 2, 5, 10$. Tables 5–8 present the empirical size $\widehat{\alpha}_{ij}, i = 1,2,3, j =$

$1, 2, 3$ in the case of $\Sigma^{(1)}$ with a uniform structure and $\Sigma^{(2)} = I$. Specifically, $\Sigma^{(1)}$ has the form

$$\Sigma^{(1)}(s,t) = \begin{cases} \sigma^2, & \text{if } s = t, \\ r\sigma^2, & \text{if } s \neq t, \end{cases} \qquad s,t = 1,\dots,p,$$

where $\sigma^2 = 5, r = 0, 0.2, 0.5, 0.9$. The sample size (N_1, N_2) is following:

$$\begin{aligned}
\text{Tables 1–2}: \quad & (10,10),\ (20,20),\ (40,40),\ (80,80), \\
& (10,20),\ (20,40), (40,80),\ (80,160), \\
& (20,10),\ (40,20),\ (80,40),\ (160,80), \\
\text{Tables 3–4}: \quad & (20,20),\ (40,40),\ (80,80),\ (160,160), \\
& (20,40),\ (40,80),\ (80,160),\ (160,320), \\
& (40,20),\ (80,40),\ (160,80),\ (320,160), \\
\text{Tables 5–6}: \quad & (10,10),\ (20,20),\ (40,40), \\
& (10,20),\ (20,40),\ (40,80), \\
& (20,10),\ (40,20),\ (80,40), \\
\text{Tables 7–8}: \quad & (20,20),\ (40,40),\ (80,80), \\
& (20,40),\ (40,80),\ (80,160), \\
& (40,20),\ (80,40),\ (160,80).
\end{aligned}$$

The last row of each table shows the average absolute discrepancy (AAD).

Tables 1–4 demonstrate that for parallelism and flatness the asymptotic approximations yield the smallest AAD in $\widehat{\alpha}_2$ and $\widehat{\alpha}_8$. In contrast, for level, the modified Bartlett-corrected approach produces more favorable AAD in $\widehat{\alpha}_6$, even when the total sample size N is relatively small. In addition, varying σ^2 has negligible impact on AAD, indicating that these methods are insensitive to changes in overall variance.

A similar trend can be observed in Tables 5–8. The asymptotic approximations remain most effective for parallelism $\widehat{\alpha}_2$ and flatness $\widehat{\alpha}_8$, while the modified Bartlett-corrected statistic for level $\widehat{\alpha}_6$ achieves the best AAD once more. Variations in the correlation coefficient ρ have a slight influence on AAD as well, suggesting that these test procedures maintain stable performance across different correlation structures.

These simulation-based results suggest that while the asymptotic approximation provides accurate approximations for the parallelism and flatness hypotheses, the modified Bartlett correction is particularly effective for the level hypothesis, especially when the sample size is small. Notably, this trend holds consistently across all scenarios considered in Tables 1–8, including settings with unequal variances and varying correlation structures, indicating that the comparative advantages of each method are insensitive to such structural variations.

4 Numerical example

In this section, we provide a numerical example of the proposed method using data from TIMSS 2019 for Japanese Grade 8 students, focusing on five subjects: mathematics, physics, chemistry, biology, and earth science. The students are first classified into two groups (non-urban and urban), with 2489 and 1927 students in each group, respectively.

Table 1: Empirical sizes($\widehat{\alpha}_1 \sim \widehat{\alpha}_9$) when $p = 3, \alpha = 0.05$, and $\sigma^2 = 2, 5, 10$

		Parallelism			Level			Flatness		
(N_1, N_2)	σ^2	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\alpha}_3$	$\widehat{\alpha}_4$	$\widehat{\alpha}_5$	$\widehat{\alpha}_6$	$\widehat{\alpha}_7$	$\widehat{\alpha}_8$	$\widehat{\alpha}_9$
(10, 10)	2	0.091	0.046	0.038	0.067	0.051	0.049	0.091	0.047	0.038
	5	0.102	0.049	0.037	0.072	0.053	0.050	0.102	0.049	0.037
	10	0.113	0.052	0.037	0.076	0.054	0.051	0.113	0.052	0.036
(20, 20)	2	0.068	0.046	0.044	0.058	0.050	0.050	0.069	0.046	0.044
	5	0.074	0.047	0.043	0.061	0.051	0.050	0.074	0.046	0.043
	10	0.078	0.046	0.041	0.063	0.051	0.050	0.078	0.047	0.042
(40, 40)	2	0.059	0.048	0.047	0.054	0.050	0.050	0.059	0.048	0.047
	5	0.061	0.047	0.046	0.055	0.050	0.050	0.061	0.047	0.046
	10	0.063	0.047	0.046	0.056	0.050	0.050	0.063	0.047	0.046
(80, 80)	2	0.054	0.048	0.048	0.051	0.050	0.049	0.054	0.049	0.049
	5	0.055	0.048	0.048	0.053	0.050	0.050	0.056	0.048	0.048
	10	0.056	0.048	0.048	0.053	0.050	0.050	0.056	0.048	0.047
AAD		2.292	0.261	0.652	0.989	0.096	0.029	2.303	0.247	0.640
(10, 20)	2	0.097	0.050	0.040	0.070	0.053	0.051	0.076	0.046	0.041
	5	0.112	0.052	0.038	0.075	0.054	0.051	0.088	0.048	0.041
	10	0.120	0.053	0.035	0.078	0.054	0.051	0.101	0.050	0.039
(20, 40)	2	0.071	0.047	0.044	0.060	0.051	0.050	0.062	0.047	0.046
	5	0.078	0.047	0.042	0.062	0.051	0.050	0.068	0.047	0.045
	10	0.081	0.047	0.041	0.064	0.051	0.050	0.073	0.047	0.043
(40, 80)	2	0.060	0.047	0.046	0.055	0.050	0.050	0.056	0.048	0.048
	5	0.064	0.047	0.046	0.056	0.051	0.050	0.059	0.048	0.047
	10	0.064	0.047	0.045	0.057	0.051	0.050	0.061	0.047	0.046
(80, 160)	2	0.055	0.049	0.048	0.052	0.050	0.050	0.053	0.049	0.049
	5	0.056	0.048	0.048	0.053	0.050	0.050	0.054	0.049	0.048
	10	0.057	0.048	0.047	0.053	0.050	0.050	0.056	0.049	0.048
AAD		2.625	0.250	0.661	1.124	0.140	0.041	1.727	0.217	0.476
(20, 10)	2	0.076	0.046	0.041	0.062	0.050	0.050	0.075	0.046	0.042
	5	0.073	0.046	0.042	0.060	0.050	0.049	0.079	0.046	0.041
	10	0.075	0.046	0.042	0.061	0.051	0.050	0.081	0.046	0.041
(40, 20)	2	0.062	0.047	0.046	0.055	0.050	0.050	0.062	0.047	0.046
	5	0.060	0.047	0.046	0.055	0.050	0.050	0.064	0.047	0.046
	10	0.062	0.047	0.046	0.056	0.051	0.050	0.065	0.047	0.045
(80, 40)	2	0.055	0.048	0.048	0.053	0.050	0.050	0.055	0.048	0.048
	5	0.056	0.049	0.048	0.052	0.050	0.050	0.057	0.048	0.048
	10	0.056	0.048	0.048	0.053	0.050	0.050	0.057	0.048	0.048
(160, 80)	2	0.053	0.049	0.049	0.051	0.050	0.050	0.053	0.050	0.050
	5	0.053	0.049	0.049	0.051	0.050	0.050	0.053	0.049	0.049
	10	0.053	0.049	0.049	0.051	0.050	0.050	0.054	0.049	0.049
AAD		1.120	0.249	0.386	0.501	0.026	0.027	1.282	0.236	0.402

Note: AAD $= \sum \big|100\hat{\alpha}_j - 100\alpha\big|/12, j = 1, 2, 3, 4, 5, 6, 7, 8, 9.$

Table 2: Empirical sizes($\widehat{\alpha}_1 \sim \widehat{\alpha}_9$) when $p = 3, \alpha = 0.01$, and $\sigma^2 = 2, 5, 10$

		Parallelism			Level			Flatness		
(N_1, N_2)	σ^2	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\alpha}_3$	$\widehat{\alpha}_4$	$\widehat{\alpha}_5$	$\widehat{\alpha}_6$	$\widehat{\alpha}_7$	$\widehat{\alpha}_8$	$\widehat{\alpha}_9$
(10, 10)	2	0.032	0.009	0.005	0.020	0.011	0.010	0.032	0.009	0.005
	5	0.039	0.011	0.005	0.023	0.012	0.010	0.039	0.011	0.005
	10	0.047	0.012	0.005	0.026	0.013	0.011	0.046	0.012	0.005
(20, 20)	2	0.019	0.009	0.007	0.015	0.010	0.010	0.019	0.009	0.007
	5	0.022	0.009	0.007	0.016	0.011	0.010	0.022	0.009	0.007
	10	0.024	0.009	0.007	0.017	0.011	0.010	0.024	0.009	0.007
(40, 40)	2	0.014	0.009	0.009	0.012	0.010	0.010	0.014	0.009	0.009
	5	0.015	0.009	0.008	0.013	0.010	0.010	0.015	0.009	0.008
	10	0.016	0.009	0.008	0.013	0.010	0.010	0.016	0.009	0.008
(80, 80)	2	0.012	0.010	0.009	0.011	0.010	0.010	0.012	0.009	0.009
	5	0.013	0.009	0.009	0.011	0.010	0.010	0.012	0.009	0.009
	10	0.013	0.009	0.009	0.012	0.010	0.010	0.013	0.009	0.009
AAD		1.216	0.107	0.252	0.567	0.071	0.020	1.217	0.106	0.253
(10, 20)	2	0.035	0.011	0.007	0.022	0.012	0.011	0.023	0.009	0.007
	5	0.046	0.012	0.006	0.025	0.013	0.011	0.030	0.010	0.007
	10	0.052	0.013	0.005	0.027	0.013	0.011	0.038	0.011	0.006
(20, 40)	2	0.021	0.009	0.008	0.015	0.011	0.010	0.016	0.009	0.008
	5	0.024	0.009	0.007	0.017	0.011	0.010	0.019	0.009	0.008
	10	0.026	0.009	0.007	0.018	0.011	0.010	0.021	0.009	0.007
(40, 80)	2	0.015	0.009	0.008	0.012	0.010	0.010	0.013	0.009	0.009
	5	0.017	0.009	0.008	0.013	0.010	0.010	0.014	0.009	0.009
	10	0.017	0.009	0.008	0.014	0.010	0.010	0.015	0.009	0.008
(80, 160)	2	0.012	0.009	0.009	0.011	0.010	0.010	0.011	0.010	0.010
	5	0.013	0.009	0.009	0.011	0.010	0.010	0.012	0.009	0.009
	10	0.013	0.009	0.009	0.012	0.010	0.010	0.013	0.009	0.009
AAD		1.422	0.126	0.249	0.646	0.100	0.033	0.871	0.085	0.187
(20, 10)	2	0.023	0.009	0.007	0.016	0.010	0.010	0.022	0.009	0.007
	5	0.022	0.008	0.007	0.016	0.010	0.010	0.025	0.009	0.007
	10	0.023	0.009	0.007	0.016	0.010	0.010	0.026	0.009	0.006
(40, 20)	2	0.016	0.009	0.008	0.013	0.010	0.010	0.016	0.009	0.008
	5	0.015	0.009	0.008	0.013	0.010	0.010	0.016	0.009	0.008
	10	0.016	0.009	0.008	0.013	0.010	0.010	0.017	0.009	0.008
(80, 40)	2	0.013	0.009	0.009	0.011	0.010	0.010	0.013	0.009	0.009
	5	0.013	0.009	0.009	0.011	0.010	0.010	0.013	0.009	0.009
	10	0.013	0.009	0.009	0.011	0.010	0.010	0.013	0.009	0.009
(160, 80)	2	0.011	0.010	0.010	0.011	0.010	0.010	0.011	0.010	0.010
	5	0.011	0.010	0.010	0.011	0.010	0.010	0.011	0.009	0.009
	10	0.011	0.009	0.009	0.011	0.010	0.010	0.012	0.009	0.009
AAD		0.540	0.097	0.164	0.270	0.013	0.010	0.624	0.092	0.172

Note: $\text{AAD} = \sum \big|100\hat{\alpha}_j - 100\alpha\big|/12, j = 1, 2, 3, 4, 5, 6, 7, 8, 9.$

Table 3: Empirical sizes($\widehat{\alpha}_1 \sim \widehat{\alpha}_9$) when $p = 8, \alpha = 0.05$, and $\sigma^2 = 2, 5, 10$

		Parallelism			Level			Flatness		
(N_1, N_2)	σ^2	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\alpha}_3$	$\widehat{\alpha}_4$	$\widehat{\alpha}_5$	$\widehat{\alpha}_6$	$\widehat{\alpha}_7$	$\widehat{\alpha}_8$	$\widehat{\alpha}_9$
(20, 20)	2	0.156	0.032	0.012	0.058	0.050	0.050	0.157	0.032	0.012
	5	0.196	0.037	0.009	0.060	0.051	0.050	0.197	0.037	0.009
	10	0.230	0.041	0.007	0.062	0.051	0.050	0.231	0.041	0.007
(40, 40)	2	0.094	0.033	0.025	0.054	0.050	0.050	0.094	0.033	0.025
	5	0.110	0.032	0.021	0.055	0.050	0.050	0.110	0.032	0.021
	10	0.122	0.031	0.018	0.056	0.051	0.050	0.122	0.032	0.018
(80, 80)	2	0.070	0.038	0.036	0.052	0.050	0.050	0.070	0.039	0.036
	5	0.077	0.037	0.033	0.053	0.050	0.050	0.076	0.036	0.032
	10	0.081	0.035	0.030	0.052	0.050	0.050	0.081	0.035	0.030
(160, 160)	2	0.059	0.043	0.043	0.051	0.050	0.050	0.060	0.044	0.043
	5	0.062	0.042	0.041	0.051	0.050	0.050	0.062	0.041	0.040
	10	0.064	0.041	0.039	0.052	0.050	0.050	0.065	0.041	0.039
AAD		6.023	1.312	2.406	0.473	0.030	0.016	6.041	1.301	2.404
(20, 40)	2	0.177	0.035	0.011	0.060	0.051	0.050	0.114	0.032	0.019
	5	0.228	0.041	0.007	0.062	0.051	0.050	0.150	0.033	0.014
	10	0.255	0.044	0.005	0.063	0.051	0.050	0.189	0.037	0.010
(40, 80)	2	0.103	0.033	0.023	0.054	0.050	0.050	0.078	0.036	0.032
	5	0.121	0.032	0.018	0.056	0.050	0.050	0.093	0.034	0.026
	10	0.131	0.032	0.016	0.056	0.050	0.050	0.109	0.033	0.022
(80, 160)	2	0.074	0.037	0.034	0.052	0.050	0.050	0.063	0.041	0.040
	5	0.082	0.036	0.030	0.053	0.050	0.050	0.070	0.039	0.036
	10	0.085	0.035	0.029	0.053	0.050	0.050	0.076	0.037	0.033
(160, 320)	2	0.061	0.042	0.041	0.051	0.050	0.050	0.056	0.045	0.045
	5	0.064	0.041	0.039	0.052	0.050	0.050	0.059	0.044	0.043
	10	0.066	0.040	0.038	0.052	0.050	0.050	0.062	0.042	0.040
AAD		7.066	1.266	2.563	0.529	0.039	0.022	4.332	1.232	1.990
(40, 20)	2	0.114	0.032	0.019	0.055	0.050	0.050	0.111	0.032	0.020
	5	0.107	0.031	0.020	0.055	0.050	0.050	0.124	0.032	0.017
	10	0.114	0.032	0.019	0.055	0.050	0.050	0.132	0.031	0.016
(80, 40)	2	0.078	0.036	0.032	0.053	0.050	0.050	0.077	0.036	0.032
	5	0.075	0.037	0.033	0.053	0.050	0.050	0.082	0.036	0.030
	10	0.078	0.036	0.032	0.053	0.050	0.050	0.084	0.034	0.028
(160, 80)	2	0.063	0.042	0.040	0.052	0.050	0.050	0.063	0.042	0.041
	5	0.062	0.042	0.041	0.051	0.050	0.050	0.065	0.041	0.039
	10	0.063	0.042	0.040	0.051	0.050	0.050	0.067	0.040	0.039
(320, 160)	2	0.056	0.045	0.044	0.051	0.050	0.050	0.056	0.045	0.045
	5	0.056	0.046	0.046	0.051	0.050	0.050	0.057	0.045	0.045
	10	0.056	0.045	0.045	0.051	0.050	0.050	0.057	0.044	0.043
AAD		2.695	1.122	1.564	0.247	0.017	0.018	3.130	1.184	1.714

Note: AAD $= \sum \big|100\hat{\alpha}_j - 100\alpha\big|/12, j = 1, 2, 3, 4, 5, 6, 7, 8, 9.$

Table 4: Empirical sizes($\widehat{\alpha}_1 \sim \widehat{\alpha}_9$) when $p = 8, \alpha = 0.01$, and $\sigma^2 = 2, 5, 10$

		Parallelism			Level			Flatness		
(N_1, N_2)	σ^2	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\alpha}_3$	$\widehat{\alpha}_4$	$\widehat{\alpha}_5$	$\widehat{\alpha}_6$	$\widehat{\alpha}_7$	$\widehat{\alpha}_8$	$\widehat{\alpha}_9$
(20, 20)	2	0.065	0.005	0.001	0.015	0.010	0.010	0.065	0.005	0.001
	5	0.091	0.006	0.000	0.016	0.011	0.010	0.092	0.007	0.000
	10	0.116	0.008	0.000	0.017	0.011	0.010	0.117	0.008	0.000
(40, 40)	2	0.029	0.005	0.003	0.012	0.010	0.010	0.029	0.005	0.003
	5	0.037	0.005	0.002	0.013	0.010	0.010	0.037	0.005	0.002
	10	0.044	0.005	0.001	0.013	0.010	0.010	0.044	0.005	0.001
(80, 80)	2	0.018	0.006	0.005	0.011	0.010	0.010	0.018	0.006	0.005
	5	0.021	0.006	0.004	0.011	0.010	0.010	0.021	0.006	0.005
	10	0.023	0.005	0.004	0.011	0.010	0.010	0.023	0.006	0.004
(160, 160)	2	0.014	0.008	0.007	0.011	0.010	0.010	0.014	0.008	0.007
	5	0.015	0.007	0.007	0.011	0.010	0.010	0.015	0.007	0.007
	10	0.015	0.007	0.006	0.011	0.010	0.010	0.015	0.007	0.006
AAD		3.052	0.388	0.654	0.259	0.022	0.009	3.066	0.387	0.654
(20, 40)	2	0.077	0.006	0.001	0.015	0.010	0.010	0.039	0.005	0.002
	5	0.114	0.008	0.000	0.017	0.011	0.010	0.060	0.005	0.001
	10	0.136	0.010	0.000	0.018	0.011	0.010	0.086	0.006	0.001
(40, 80)	2	0.033	0.005	0.002	0.012	0.010	0.010	0.021	0.006	0.004
	5	0.044	0.005	0.001	0.013	0.010	0.010	0.028	0.005	0.003
	10	0.049	0.005	0.001	0.013	0.010	0.010	0.036	0.005	0.002
(80, 160)	2	0.019	0.006	0.005	0.011	0.010	0.010	0.015	0.007	0.007
	5	0.023	0.005	0.004	0.011	0.010	0.010	0.017	0.006	0.005
	10	0.025	0.005	0.003	0.012	0.010	0.010	0.020	0.006	0.005
(160, 320)	2	0.014	0.007	0.007	0.011	0.010	0.010	0.012	0.008	0.008
	5	0.016	0.007	0.006	0.011	0.010	0.010	0.013	0.008	0.007
	10	0.016	0.007	0.006	0.011	0.010	0.010	0.014	0.007	0.007
AAD		3.710	0.372	0.687	0.290	0.024	0.008	2.030	0.381	0.572
(40, 20)	2	0.040	0.005	0.002	0.013	0.010	0.010	0.038	0.005	0.002
	5	0.035	0.005	0.002	0.013	0.010	0.010	0.045	0.005	0.001
	10	0.040	0.005	0.002	0.013	0.010	0.010	0.049	0.005	0.001
(80, 40)	2	0.021	0.006	0.004	0.011	0.010	0.010	0.021	0.006	0.004
	5	0.020	0.006	0.005	0.011	0.010	0.010	0.023	0.006	0.004
	10	0.021	0.006	0.004	0.011	0.010	0.010	0.024	0.005	0.004
(160, 80)	2	0.015	0.007	0.007	0.011	0.010	0.010	0.015	0.007	0.007
	5	0.014	0.007	0.007	0.011	0.010	0.010	0.016	0.007	0.006
	10	0.015	0.007	0.007	0.011	0.010	0.010	0.016	0.007	0.006
(320, 160)	2	0.012	0.008	0.008	0.010	0.010	0.010	0.012	0.008	0.008
	5	0.012	0.008	0.008	0.010	0.010	0.010	0.013	0.008	0.008
	10	0.012	0.008	0.008	0.010	0.010	0.010	0.013	0.008	0.008
AAD		1.147	0.352	0.474	0.129	0.008	0.008	1.370	0.369	0.509

Note: AAD $= \sum \big|100\hat{\alpha}_j - 100\alpha\big|/12, j = 1,2,3,4,5,6,7,8,9.$

Table 5: Empirical sizes($\widehat{\alpha}_1 \sim \widehat{\alpha}_9$) when $p = 3, \alpha = 0.05$, and $r = 0, 0.2, 0.5, 0.9$

		Parallelism			Level			Flatness		
(N_1, N_2)	ρ	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\alpha}_3$	$\widehat{\alpha}_4$	$\widehat{\alpha}_5$	$\widehat{\alpha}_6$	$\widehat{\alpha}_7$	$\widehat{\alpha}_8$	$\widehat{\alpha}_9$
(10, 10)	0	0.102	0.049	0.037	0.072	0.053	0.050	0.102	0.049	0.037
	0.2	0.099	0.049	0.037	0.074	0.053	0.050	0.100	0.049	0.038
	0.5	0.093	0.047	0.038	0.075	0.054	0.050	0.093	0.047	0.038
	0.9	0.090	0.046	0.038	0.077	0.054	0.050	0.091	0.046	0.037
(20, 20)	0	0.074	0.047	0.043	0.061	0.051	0.050	0.074	0.046	0.043
	0.2	0.072	0.046	0.043	0.061	0.051	0.050	0.073	0.047	0.043
	0.5	0.070	0.047	0.044	0.062	0.051	0.050	0.070	0.046	0.044
	0.9	0.069	0.047	0.044	0.063	0.051	0.050	0.068	0.046	0.044
(40, 40)	0	0.061	0.047	0.046	0.055	0.050	0.050	0.061	0.047	0.046
	0.2	0.061	0.048	0.047	0.056	0.050	0.050	0.061	0.047	0.046
	0.5	0.059	0.047	0.047	0.056	0.050	0.050	0.059	0.047	0.047
	0.9	0.059	0.047	0.047	0.056	0.050	0.050	0.059	0.047	0.047
AAD		2.584	0.278	0.752	1.402	0.152	0.021	2.584	0.283	0.755
(10, 20)	0	0.112	0.052	0.038	0.075	0.054	0.051	0.088	0.048	0.041
	0.2	0.109	0.052	0.038	0.077	0.054	0.051	0.085	0.047	0.041
	0.5	0.101	0.050	0.039	0.078	0.054	0.050	0.078	0.046	0.041
	0.9	0.076	0.046	0.041	0.079	0.054	0.050	0.075	0.046	0.042
(20, 40)	0	0.078	0.047	0.042	0.062	0.051	0.050	0.068	0.047	0.045
	0.2	0.076	0.046	0.042	0.063	0.051	0.050	0.066	0.047	0.045
	0.5	0.074	0.047	0.044	0.063	0.051	0.050	0.063	0.047	0.045
	0.9	0.062	0.047	0.046	0.064	0.051	0.050	0.062	0.047	0.046
(40, 80)	0	0.064	0.047	0.046	0.056	0.051	0.050	0.059	0.048	0.047
	0.2	0.063	0.047	0.046	0.057	0.050	0.050	0.057	0.047	0.047
	0.5	0.061	0.047	0.046	0.057	0.051	0.050	0.056	0.048	0.048
	0.9	0.056	0.048	0.047	0.057	0.051	0.050	0.055	0.048	0.048
AAD		2.752	0.268	0.703	1.567	0.195	0.032	1.765	0.282	0.536
(20, 10)	0	0.073	0.046	0.042	0.060	0.050	0.049	0.079	0.046	0.041
	0.2	0.073	0.045	0.042	0.060	0.050	0.049	0.078	0.046	0.042
	0.5	0.074	0.045	0.041	0.061	0.050	0.050	0.075	0.046	0.042
	0.9	0.097	0.050	0.040	0.062	0.051	0.050	0.077	0.046	0.042
(40, 20)	0	0.060	0.047	0.046	0.055	0.050	0.050	0.064	0.047	0.046
	0.2	0.061	0.047	0.046	0.055	0.050	0.050	0.063	0.047	0.046
	0.5	0.061	0.047	0.046	0.055	0.050	0.050	0.062	0.047	0.046
	0.9	0.072	0.047	0.044	0.056	0.050	0.050	0.062	0.047	0.046
(80, 40)	0	0.056	0.049	0.048	0.052	0.050	0.050	0.057	0.048	0.048
	0.2	0.055	0.048	0.048	0.052	0.050	0.050	0.056	0.048	0.048
	0.5	0.055	0.048	0.048	0.053	0.050	0.050	0.056	0.048	0.048
	0.9	0.060	0.047	0.046	0.053	0.050	0.050	0.056	0.048	0.048
AAD		1.647	0.288	0.527	0.619	0.023	0.027	1.523	0.297	0.501

Note: $\text{AAD} = \sum \big|100\hat{\alpha}_j - 100\alpha\big|/12, j = 1, 2, 3, 4, 5, 6, 7, 8, 9.$

Table 6: Empirical sizes($\widehat{\alpha}_1 \sim \widehat{\alpha}_9$) when $p = 3, \alpha = 0.01$, and $r = 0, 0.2, 0.5, 0.9$

		Parallelism			Level			Flatness		
(N_1, N_2)	ρ	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\alpha}_3$	$\widehat{\alpha}_4$	$\widehat{\alpha}_5$	$\widehat{\alpha}_6$	$\widehat{\alpha}_7$	$\widehat{\alpha}_8$	$\widehat{\alpha}_9$
(10, 10)	0	0.039	0.011	0.005	0.023	0.012	0.010	0.039	0.011	0.005
	0.2	0.037	0.010	0.005	0.024	0.013	0.011	0.038	0.010	0.005
	0.5	0.033	0.009	0.005	0.026	0.013	0.011	0.033	0.009	0.005
	0.9	0.032	0.009	0.005	0.026	0.013	0.010	0.032	0.009	0.005
(20, 20)	0	0.022	0.009	0.007	0.016	0.011	0.010	0.022	0.009	0.007
	0.2	0.021	0.009	0.007	0.016	0.011	0.010	0.021	0.009	0.007
	0.5	0.019	0.009	0.007	0.017	0.011	0.010	0.020	0.009	0.007
	0.9	0.019	0.009	0.007	0.017	0.011	0.010	0.019	0.009	0.007
(40, 40)	0	0.015	0.009	0.008	0.013	0.010	0.010	0.015	0.009	0.008
	0.2	0.015	0.009	0.008	0.013	0.010	0.010	0.015	0.009	0.008
	0.5	0.014	0.009	0.009	0.013	0.010	0.010	0.014	0.009	0.009
	0.9	0.014	0.009	0.009	0.013	0.010	0.010	0.014	0.009	0.009
AAD		1.351	0.099	0.297	0.812	0.113	0.023	1.352	0.100	0.296
(10, 20)	0	0.046	0.012	0.006	0.025	0.013	0.011	0.030	0.010	0.007
	0.2	0.044	0.012	0.006	0.026	0.013	0.011	0.028	0.010	0.007
	0.5	0.038	0.011	0.006	0.027	0.013	0.011	0.024	0.009	0.007
	0.9	0.023	0.009	0.006	0.028	0.013	0.010	0.022	0.009	0.007
(20, 40)	0	0.024	0.009	0.007	0.017	0.011	0.010	0.019	0.009	0.008
	0.2	0.023	0.009	0.007	0.017	0.011	0.010	0.018	0.009	0.008
	0.5	0.022	0.009	0.007	0.017	0.011	0.010	0.016	0.009	0.008
	0.9	0.016	0.009	0.008	0.018	0.011	0.010	0.016	0.009	0.008
(40, 80)	0	0.017	0.009	0.008	0.013	0.010	0.010	0.014	0.009	0.009
	0.2	0.016	0.009	0.008	0.013	0.010	0.010	0.013	0.009	0.009
	0.5	0.015	0.009	0.008	0.014	0.010	0.010	0.013	0.009	0.009
	0.9	0.012	0.009	0.009	0.014	0.010	0.010	0.012	0.009	0.009
AAD		1.462	0.128	0.270	0.914	0.140	0.032	0.879	0.093	0.213
(20, 10)	0	0.022	0.008	0.007	0.016	0.010	0.010	0.025	0.009	0.007
	0.2	0.022	0.008	0.007	0.016	0.010	0.010	0.024	0.009	0.007
	0.5	0.022	0.008	0.007	0.016	0.010	0.010	0.023	0.009	0.007
	0.9	0.036	0.011	0.007	0.017	0.011	0.010	0.023	0.009	0.007
(40, 20)	0	0.015	0.009	0.008	0.013	0.010	0.010	0.016	0.009	0.008
	0.2	0.015	0.009	0.008	0.013	0.010	0.010	0.016	0.009	0.008
	0.5	0.015	0.009	0.008	0.013	0.010	0.010	0.016	0.009	0.008
	0.9	0.020	0.009	0.007	0.013	0.010	0.010	0.016	0.009	0.008
(80, 40)	0	0.013	0.009	0.009	0.011	0.010	0.010	0.013	0.009	0.009
	0.2	0.012	0.009	0.009	0.011	0.010	0.010	0.013	0.009	0.009
	0.5	0.012	0.009	0.009	0.011	0.010	0.010	0.012	0.009	0.009
	0.9	0.015	0.009	0.009	0.011	0.010	0.010	0.013	0.009	0.009
AAD		0.824	0.110	0.209	0.338	0.019	0.010	0.747	0.113	0.211

Note: AAD $= \sum \left| 100\hat{\alpha}_j - 100\alpha \right| / 12, j = 1, 2, 3, 4, 5, 6, 7, 8, 9.$

Table 7: Empirical sizes($\widehat{\alpha}_1 \sim \widehat{\alpha}_9$) when $p = 8, \alpha = 0.05$, and $r = 0, 0.2, 0.5, 0.9$

		Parallelism			Level			Flatness		
(N_1, N_2)	ρ	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\alpha}_3$	$\widehat{\alpha}_4$	$\widehat{\alpha}_5$	$\widehat{\alpha}_6$	$\widehat{\alpha}_7$	$\widehat{\alpha}_8$	$\widehat{\alpha}_9$
(20, 20)	0	0.196	0.037	0.009	0.060	0.051	0.050	0.197	0.037	0.009
	0.2	0.256	0.045	0.005	0.064	0.051	0.050	0.256	0.045	0.005
	0.5	0.240	0.042	0.006	0.065	0.051	0.050	0.239	0.042	0.006
	0.9	0.165	0.033	0.011	0.064	0.051	0.050	0.165	0.033	0.011
(40, 40)	0	0.110	0.032	0.021	0.055	0.050	0.050	0.110	0.032	0.021
	0.2	0.132	0.032	0.016	0.057	0.050	0.050	0.131	0.032	0.016
	0.5	0.125	0.031	0.017	0.057	0.050	0.050	0.125	0.032	0.017
	0.9	0.098	0.033	0.024	0.057	0.050	0.050	0.098	0.033	0.024
(80, 80)	0	0.077	0.037	0.033	0.053	0.050	0.050	0.076	0.036	0.032
	0.2	0.085	0.035	0.029	0.053	0.049	0.049	0.085	0.035	0.029
	0.5	0.083	0.035	0.030	0.054	0.050	0.050	0.083	0.035	0.030
	0.9	0.072	0.038	0.035	0.054	0.050	0.050	0.071	0.038	0.035
AAD		8.633	1.423	3.038	0.769	0.050	0.013	8.632	1.423	3.044
(20, 40)	0	0.228	0.041	0.007	0.062	0.051	0.050	0.150	0.033	0.014
	0.2	0.273	0.047	0.004	0.065	0.051	0.050	0.226	0.041	0.007
	0.5	0.262	0.046	0.005	0.065	0.051	0.050	0.202	0.038	0.009
	0.9	0.190	0.037	0.010	0.065	0.052	0.050	0.121	0.032	0.018
(40, 80)	0	0.121	0.032	0.018	0.056	0.050	0.050	0.093	0.034	0.026
	0.2	0.136	0.031	0.014	0.057	0.050	0.050	0.121	0.032	0.018
	0.5	0.133	0.032	0.015	0.057	0.050	0.050	0.113	0.032	0.020
	0.9	0.108	0.033	0.022	0.057	0.050	0.050	0.081	0.036	0.031
(80, 160)	0	0.082	0.036	0.030	0.053	0.050	0.050	0.070	0.039	0.036
	0.2	0.087	0.034	0.028	0.053	0.050	0.050	0.081	0.036	0.030
	0.5	0.086	0.035	0.028	0.054	0.050	0.050	0.078	0.036	0.032
	0.9	0.076	0.037	0.033	0.053	0.050	0.050	0.065	0.041	0.040
AAD		9.845	1.332	3.203	0.806	0.055	0.015	6.679	1.424	2.647
(40, 20)	0	0.107	0.031	0.020	0.055	0.050	0.050	0.124	0.032	0.017
	0.2	0.124	0.032	0.017	0.057	0.050	0.050	0.137	0.031	0.015
	0.5	0.117	0.031	0.019	0.057	0.050	0.050	0.134	0.031	0.015
	0.9	0.110	0.032	0.020	0.056	0.050	0.050	0.114	0.032	0.020
(80, 40)	0	0.075	0.037	0.033	0.053	0.050	0.050	0.082	0.036	0.030
	0.2	0.082	0.035	0.030	0.053	0.050	0.050	0.087	0.034	0.028
	0.5	0.079	0.036	0.031	0.054	0.050	0.050	0.086	0.034	0.028
	0.9	0.076	0.036	0.032	0.054	0.050	0.050	0.078	0.036	0.032
(160, 80)	0	0.062	0.042	0.041	0.051	0.050	0.050	0.065	0.041	0.039
	0.2	0.065	0.041	0.039	0.052	0.050	0.050	0.067	0.040	0.038
	0.5	0.063	0.041	0.040	0.052	0.050	0.050	0.066	0.040	0.038
	0.9	0.063	0.042	0.041	0.051	0.050	0.050	0.063	0.042	0.040
AAD		3.526	1.361	1.966	0.367	0.024	0.022	4.193	1.425	2.170

Note: AAD $= \sum \left|100\hat{\alpha}_j - 100\alpha\right|/12, j = 1, 2, 3, 4, 5, 6, 7, 8, 9.$

Table 8: Empirical sizes($\widehat{\alpha}_1 \sim \widehat{\alpha}_9$) when $p = 8, \alpha = 0.01$, and $r = 0, 0.2, 0.5, 0.9$

		Parallelism			Level			Flatness		
(N_1, N_2)	ρ	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\alpha}_3$	$\widehat{\alpha}_4$	$\widehat{\alpha}_5$	$\widehat{\alpha}_6$	$\widehat{\alpha}_7$	$\widehat{\alpha}_8$	$\widehat{\alpha}_9$
(20, 20)	0	0.091	0.006	0.000	0.016	0.011	0.010	0.092	0.007	0.000
	0.2	0.137	0.010	0.000	0.018	0.011	0.010	0.137	0.010	0.000
	0.5	0.124	0.009	0.000	0.018	0.011	0.010	0.123	0.009	0.000
	0.9	0.070	0.005	0.001	0.018	0.011	0.010	0.070	0.005	0.001
(40, 40)	0	0.037	0.005	0.002	0.013	0.010	0.010	0.037	0.005	0.002
	0.2	0.050	0.005	0.001	0.014	0.010	0.010	0.049	0.005	0.001
	0.5	0.045	0.005	0.001	0.014	0.010	0.010	0.046	0.005	0.001
	0.9	0.031	0.005	0.003	0.014	0.010	0.010	0.030	0.005	0.002
(80, 80)	0	0.021	0.006	0.004	0.011	0.010	0.010	0.021	0.006	0.005
	0.2	0.024	0.005	0.004	0.012	0.010	0.010	0.024	0.005	0.004
	0.5	0.024	0.005	0.004	0.012	0.010	0.010	0.023	0.005	0.004
	0.9	0.018	0.006	0.005	0.012	0.010	0.010	0.018	0.006	0.005
AAD		4.592	0.400	0.788	0.429	0.039	0.007	4.591	0.401	0.789
(20, 40)	0	0.114	0.008	0.000	0.017	0.011	0.010	0.060	0.005	0.001
	0.2	0.151	0.011	0.000	0.018	0.011	0.010	0.113	0.008	0.000
	0.5	0.142	0.010	0.000	0.018	0.011	0.010	0.095	0.007	0.000
	0.9	0.086	0.006	0.001	0.018	0.011	0.010	0.043	0.005	0.002
(40, 80)	0	0.044	0.005	0.001	0.013	0.010	0.010	0.028	0.005	0.003
	0.2	0.052	0.005	0.001	0.014	0.010	0.010	0.043	0.005	0.002
	0.5	0.051	0.005	0.001	0.014	0.010	0.010	0.039	0.005	0.002
	0.9	0.036	0.005	0.002	0.014	0.010	0.010	0.022	0.005	0.004
(80, 160)	0	0.023	0.005	0.004	0.011	0.010	0.010	0.017	0.006	0.005
	0.2	0.025	0.005	0.003	0.012	0.010	0.010	0.023	0.005	0.004
	0.5	0.025	0.005	0.003	0.012	0.010	0.010	0.021	0.006	0.004
	0.9	0.020	0.006	0.005	0.012	0.010	0.010	0.015	0.007	0.006
AAD		5.404	0.380	0.816	0.446	0.037	0.008	3.331	0.424	0.720
(40, 20)	0	0.035	0.005	0.002	0.013	0.010	0.010	0.045	0.005	0.001
	0.2	0.045	0.005	0.001	0.014	0.010	0.010	0.052	0.005	0.001
	0.5	0.041	0.005	0.002	0.014	0.010	0.010	0.051	0.005	0.001
	0.9	0.037	0.005	0.002	0.014	0.010	0.010	0.039	0.005	0.002
(80, 40)	0	0.020	0.006	0.005	0.011	0.010	0.010	0.023	0.006	0.004
	0.2	0.023	0.005	0.004	0.012	0.010	0.010	0.025	0.005	0.003
	0.5	0.022	0.005	0.004	0.012	0.010	0.010	0.025	0.005	0.003
	0.9	0.020	0.006	0.004	0.012	0.010	0.010	0.021	0.005	0.004
(160, 80)	0	0.014	0.007	0.007	0.011	0.010	0.010	0.016	0.007	0.006
	0.2	0.015	0.007	0.006	0.011	0.010	0.010	0.016	0.007	0.006
	0.5	0.015	0.007	0.007	0.011	0.010	0.010	0.016	0.007	0.006
	0.9	0.015	0.007	0.007	0.011	0.010	0.010	0.015	0.007	0.007
AAD		1.526	0.420	0.584	0.196	0.012	0.009	1.881	0.435	0.626

Note: AAD $= \sum \left|100\hat{\alpha}_j - 100\alpha\right|/12, j = 1, 2, 3, 4, 5, 6, 7, 8, 9.$

After excluding outliers beyond three standard deviations, the remaining sample sizes are 2468 in the non-urban group and 1915 in the urban group. We confirm that the data could be reasonably treated as normally distributed. The covariance matrices of the urban and non-urban groups are found to differ, thus supporting the application of the proposed test for this dataset. To illustrate its application, we randomly selected 40 individuals from each group and conducted a profile analysis. Figure 1 displays the mean profiles of the urban and non-urban groups across the five subjects. This plot shows the differences in the profile patterns between the two groups. In this example,

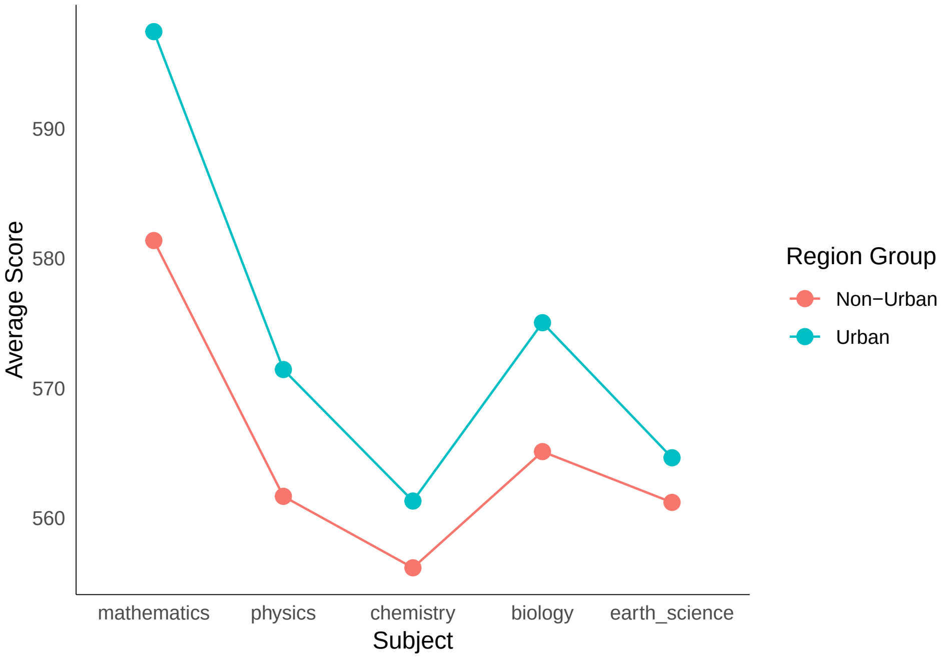

Figure 1: Mean subject profiles by region group

the three profile hypotheses are evaluated with the asymptotic expansion of the upper critical value, where its limiting value is given by the chi-squared distribution, and the modified Bartlett-corrected statistic. For the parallelism hypothesis, the observed test statistic is $T_1^2 = 1.68$, which is sufficiently below both the critical value from the chi-squared distribution, $\chi_4^2(0.05) = 9.49$, and the value from the asymptotic expansion, $\widehat{t_1^2}(0.05) = 10.89$. The modified Bartlett-corrected statistic is $T_{1B}^2 = 1.60$, which has a value smaller than the chi-square critical value as well. Therefore, the parallelism hypothesis is not rejected in any of the three cases. For the level hypothesis, a similar result is obtained, where the test statistic is $T_2^2 = 0.27$, while the critical values are $\chi_1^2(0.05) = 3.84$ from the chi-squared distribution and $\widehat{t_2^2}(0.05) = 3.96$ computed from the asymptotic expansion. The modified Bartlett-corrected value remains $T_{2B}^2 = 0.27$, leading to no rejection of the hypothesis under any approach. In contrast, for the flatness hypothesis, the test statistic is $T_3^2 = 42.47$, which greatly exceeds both the chi-square critical value $\chi_4^2(0.05) = 9.49$ and the asymptotic expansion value $\widehat{t_3^2}(0.05) = 10.89$. Even after applying the modified Bartlett correction, the adjusted value remains large at $T_{3B}^2 = 28.0$, resulting in rejection of the flatness hypothesis under all three cases. While all three decision rules yield identical conclusions in the present example, their performance may differ depending on sample size and dimensionality. It is therefore recommended that practitioners refer to the simulation results presented in Section 3 and choose the criterion best suited to the specific structure of their data.

The results indicate that the profiles of the two groups are not significantly different in terms of parallelism and level, but both profiles deviate from flatness. From these findings, we conclude that the newly proposed statistic can be applied to real datasets such as TIMSS 2019, and that the test procedures perform appropriately under conditions in which the covariance structures differ between groups.

5 Conclusion

This study considered profile analysis under the assumption of unequal covariance matrices. We proposed test statistics for testing parallelism, level, and flatness hypotheses in a two-sample problem and derived the approximate upper percentiles of these test statistics. In addition, we proposed modified Bartlett corrected statistics for three hypotheses. The approximate upper percentiles presented in this paper have good approximation. Based on the simulation results for empirical sizes, the asymptotic expansion approach provides accurate approximations for both the parallelism and flatness hypotheses. In contrast, for the level hypothesis, the modified Bartlett-corrected statistic yields better approximations. Furthermore, modifying the variance structure does not significantly diminish accuracy, indicating that the proposed methods are insensitive to changes in the variance structure.

Appendix

A.1 Expand T_1^2 by perturbation method

Let

$$\rho_i = \sqrt{\frac{N_i - 1}{N - 2}}, \quad \Omega_{1i} = \frac{1}{\sqrt{N_i}}(C\overline{\Sigma}_1 C')^{-\frac{1}{2}}(C\Sigma^{(i)}C')^{\frac{1}{2}},$$
$$V_1^{(i)} = \sqrt{N_i - 1}\left[(C\Sigma^{(i)}C')^{-\frac{1}{2}}CS^{(i)}C'(C\Sigma^{(i)}C')^{-\frac{1}{2}} - I_{p-1}\right], \quad i = 1, 2.$$

Then, W^{-1} can be expanded as

$$W^{-1} = I_{p-1} - \frac{1}{\sqrt{n}}V_1 + \frac{1}{n}V_1^2 + O_p(n^{-\frac{3}{2}}),$$

where

$$n = N - 2, \quad V_1 = \sum_{i=1}^{2} \rho_i^{-1}\Omega_{1i}V_1^{(i)}\Omega_{1i}'.$$

Note that V_1 and $\boldsymbol{z}$ are independent. Therefore, using W^{-1}, T_1^2 can be expressed as

$$T_1^2 = \boldsymbol{z}_1'\boldsymbol{z}_1 - \frac{1}{\sqrt{n}}\boldsymbol{z}_1'V_1\boldsymbol{z}_1 + \frac{1}{n}\boldsymbol{z}_1'V_1^2\boldsymbol{z}_1 + O_p(n^{-\frac{3}{2}}),$$

where $\boldsymbol{z}_1$ is independently normally distributed as $N_{p-1}(\mathbf{0}, I_{p-1})$.

A.2 Asymptotic expansion of the characteristic function of T_1^2

The asymptotic expansion for the characteristic function of T_1^2 can be derived using the above results.

$$
\begin{aligned}
&E[\exp(itT_1^2)] \\
&= \underbrace{E[\exp(it\boldsymbol{z}_1'\boldsymbol{z}_1)]}_{(1)} + \frac{1}{\sqrt{n}}\underbrace{E[(-it)\boldsymbol{z}_1'V_1\boldsymbol{z}_1\exp(it\boldsymbol{z}_1'\boldsymbol{z}_1)]}_{(2)} + \frac{1}{n}\underbrace{E[it\boldsymbol{z}_1'V_1^2\boldsymbol{z}_1\exp(it\boldsymbol{z}_1'\boldsymbol{z}_1)]}_{(3)} \\
&+ \frac{1}{2n}\underbrace{E[(it)^2(\boldsymbol{z}_1'V_1\boldsymbol{z}_1)^2\exp(it\boldsymbol{z}_1'\boldsymbol{z}_1)]}_{(4)} + O_p(n^{-\frac{3}{2}}).
\end{aligned}
$$

Since $\boldsymbol{z}_1'\boldsymbol{z}_1 \sim \chi^2_{p-1}$, we have

$$(1) = (1-2it)^{-\frac{p-1}{2}}.$$

Since $E(V_1) = 0$, we get

$$(2) = 0.$$

(3) and (4) are obtained as follows:

$$
\begin{aligned}
(3) &= it(a_1+b_1)(1-2it)^{-\frac{p-1}{2}-1}, \\
(4) &= 2(it)^2(a_1+2b_1)^{-\frac{p-1}{2}-2},
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \rho_1^{-2}(\mathrm{tr}\Omega_{11}\Omega_{11}')^2 + \rho_2^{-2}(\mathrm{tr}\Omega_{12}\Omega_{12}')^2, \\
b_1 &= \rho_1^{-2}\mathrm{tr}(\Omega_{11}\Omega_{11}')^2 + \rho_2^{-2}\mathrm{tr}(\Omega_{12}\Omega_{12}')^2.
\end{aligned}
$$

Therefore, the characteristic function of T_1^2 is

$$E[\exp(itT_1^2)] = u^{-\frac{p-1}{2}} + \frac{1}{n}\sum_{j=0}^{2}\beta_{1j}u^{-\frac{p-1}{2}-j} + O(n^{-\frac{3}{2}}),$$

where

$$u = 1-2it, \quad \beta_{10} = -\frac{1}{4}a_1, \quad \beta_{11} = -\frac{1}{2}b_1, \quad \beta_{12} = \frac{1}{4}(a_1+2b_1).$$

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