T^2-type Test Statistic for Testing the Adequacy in Growth Curve Model with Two-step Monotone Missing Data

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Abstract

This article considers a test for the adequacy of a growth curve model when the data set has a two-step monotone pattern of missing observations. For complete data, several studies have employed a Hotelling T^2-type test statistic. In addition, T^2-type tests for mean vectors with monotone missing data have been developed and investigated. We extend these results to the problem of testing the adequacy of a growth curve model with missing values. Specifically, we propose a T^2-type test statistic for the growth curve model with two-step monotone missing data. Because deriving the exact null distribution of the proposed test statistic is difficult, we provide an approximate upper percentile of its null distribution. Finally, we numerically evaluate the accuracy of this approximation using Monte Carlo simulation.

Key Words and Phrases: F-approximation, Maximum likelihood estimator, Monte Carlo simulation

1 Introduction

This study considers a test for adequacy $H_0 : \boldsymbol{\mu} = \boldsymbol{X\theta}$ vs. $H_1 : \boldsymbol{\mu} \neq \boldsymbol{X\theta}$ in growth curve model with two-step monotone missing data, where $\boldsymbol{\mu}$ denotes the mean vector, $\boldsymbol{X}$ is within-design matrix, and $\boldsymbol{\theta}$ is an unknown parameter vector. A growth curve model describes measurements collected repeatedly over time for each individual and assumes the structure $\boldsymbol{X\theta}$ for the mean vector $\boldsymbol{\mu}$. Maximum likelihood estimators (MLEs) of the normal mean vector and covariance matrix under monotone missing data, without structural assumptions, were derived by Kanda and Fujikoshi (1998) and Yagi and Seo (2017). MLEs under a growth curve model were subsequently derived by Yagi, Seo and Fujikoshi (2021). The tests for normal mean vectors under two-step monotone missing data have also been discussed by Seko, Yamazaki, and Seo (2012), Kawasaki and Seo (2016) and Kawasaki, Shutoh, and Seo (2018). The cases

of three-step and general k-step monotone missing data have been investigated by Krishnamoorthy and Pannnala (1999), Yagi and Seo (2017) and Yagi, Seo, and Hanusz (2019).

For testing the adequacy of a growth curve model with complete data, Fujikoshi, Ulyanov, and Shimizu (2010) proposed a Hotelling-type test statistic and derived its null distribution. First, we introduce the test of the hypothesis for adequacy with complete data in growth curve model. Let $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_N$ be distributed as the multivariate normal $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and define the sample mean vector $\overline{\boldsymbol{y}}$ and the sample covariance matrix $\boldsymbol{S}$ as follows:

$$\overline{\boldsymbol{y}} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{y}_i, \quad \boldsymbol{S} = \frac{1}{N-1}\sum_{i=1}^{N} (\boldsymbol{y}_i - \overline{\boldsymbol{y}})(\boldsymbol{y}_i - \overline{\boldsymbol{y}})'.$$

We consider a test of the hypothesis $H_0 : \boldsymbol{\mu} = \boldsymbol{X}\boldsymbol{\theta}$ for the mean structure. The alternative hypothesis $H_1 : \boldsymbol{\mu} \neq \boldsymbol{X}\boldsymbol{\theta}$ is an unconstrained, unstructured model. Let $p \times (p-q)$ matrix $\widetilde{\boldsymbol{X}}$ satisfies $\widetilde{\boldsymbol{X}}'\boldsymbol{X} = \boldsymbol{O}$ and $\widetilde{\boldsymbol{X}}'\widetilde{\boldsymbol{X}} = \boldsymbol{I}_{p-q}$, which satisfies the following equation:

$$\boldsymbol{I}_p - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X} = \widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{X}}'.$$

The hypothesis for the mean structure is equivalent to $H_0 : \widetilde{\boldsymbol{X}}'\boldsymbol{\mu} = \boldsymbol{0}$ because

$$\widetilde{\boldsymbol{X}}'\boldsymbol{\mu} = \widetilde{\boldsymbol{X}}'\boldsymbol{X}\boldsymbol{\theta} = \boldsymbol{0}.$$

Let $\boldsymbol{z} = \widetilde{\boldsymbol{X}}'\overline{\boldsymbol{y}}$. As $\overline{\boldsymbol{y}} \sim N_p(\boldsymbol{\mu}, N^{-1}\boldsymbol{\Sigma})$,

$$\boldsymbol{z} \sim N_{p-q}\Big(\widetilde{\boldsymbol{X}}'\boldsymbol{\mu}, \frac{1}{N}\widetilde{\boldsymbol{X}}'\boldsymbol{\Sigma}\widetilde{\boldsymbol{X}}\Big).$$

$N^{-1}\widetilde{\boldsymbol{X}}'\boldsymbol{\Sigma}\widetilde{\boldsymbol{X}}$ can be estimated by $N^{-1}\widetilde{\boldsymbol{X}}'\boldsymbol{S}\widetilde{\boldsymbol{X}}$. Subsequently,

$$\sqrt{N}\boldsymbol{z} \sim N_{p-q}(\sqrt{N}\widetilde{\boldsymbol{X}}'\boldsymbol{\mu}, \widetilde{\boldsymbol{X}}'\boldsymbol{\Sigma}\widetilde{\boldsymbol{X}}),$$
$$(N-1)\widetilde{\boldsymbol{X}}'\boldsymbol{S}\widetilde{\boldsymbol{X}} \sim W_{p-q}(N-1, \widetilde{\boldsymbol{X}}'\boldsymbol{\Sigma}\widetilde{\boldsymbol{X}})$$

and they are independent of each other. Therefore, the test for adequacy with complete data can be tested using a Hotelling T^2-type test statistic

$$T^2 = (\sqrt{N}\boldsymbol{z})'(\widetilde{\boldsymbol{X}}'\boldsymbol{S}\widetilde{\boldsymbol{X}})^{-1}(\sqrt{N}\boldsymbol{z}).$$

It holds that

$$T^2 \sim \frac{(N-1)(p-q)}{N-(p-q)} \cdot F_{p-q,N-(p-q)}$$

under the hypothesis H_0, where $F_{p-q,\ N-(p-q)}$ is the F-distribution with degree of freedom $p-q$ and $N-(p-q)$.

In this study, we propose a test statistic for adequacy in growth curve model with two-step monotone missing data and produce an approximate upper percentile for its null distribution.

The remainder of this paper is organized as follows. Section 2 describes two-step monotone missing data and presents the MLEs of the mean vector and covariance matrix in this setting. Section 3 discusses the test statistic for adequacy and its null distribution under two-step monotone missing data. Section 4 presents the simulation results. Finally, Section 5 concludes the paper.

2 Two-step monotone missing data

Let $\boldsymbol{y}_1^{(1)}, \boldsymbol{y}_2^{(1)}, \ldots, \boldsymbol{y}_{N_1}^{(1)}$ be distributed as the multivariate normal $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{y}_1^{(2)}, \boldsymbol{y}_2^{(2)}, \ldots, \boldsymbol{y}_{N_2}^{(2)}$ be distributed as the multivariate normal $N_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$, where each $\boldsymbol{y}_j^{(1)} = (y_{j1}^{(1)}, y_{j2}^{(1)}, \ldots, y_{jp}^{(1)})'$, $j = 1, 2, \ldots, N_1$ is $p \times 1$ and $\boldsymbol{y}_j^{(2)} = (y_{j1}^{(2)}, y_{j2}^{(2)}, \ldots, y_{jp_1}^{(2)})'$, $j = 1, 2, \ldots, N_2$ is $p_1 \times 1$, and

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

In the growth curve model, the mean vector $\boldsymbol{\mu}$ is

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{pmatrix} \boldsymbol{\theta} = \boldsymbol{X}\boldsymbol{\theta},$$

where $\boldsymbol{X}$ is a denoted $p \times q$ matrix with rank q, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_q)'$ is an unknown parameter vector, and $\boldsymbol{\Sigma}$ is unknown positive definite. The matrix $\boldsymbol{X}$ is called a within-design matrix. Specifically, two-step monotone missing data is drawn from a multivariate normal population of the form

$$\begin{pmatrix} y_{11}^{(1)} & y_{12}^{(1)} & \cdots & y_{1p_1}^{(1)} & y_{1p_1+1}^{(1)} & \cdots & y_{1p}^{(1)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ y_{N_1 1}^{(1)} & y_{N_1 2}^{(1)} & \cdots & y_{N_1 p_1}^{(1)} & y_{N_1 p_1+1}^{(1)} & \cdots & y_{N_1 p}^{(1)} \\ y_{11}^{(2)} & y_{12}^{(2)} & \cdots & y_{1p_1}^{(2)} & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ y_{N_2 1}^{(2)} & y_{N_2 2}^{(2)} & \cdots & y_{N_2 p_1}^{(2)} & * & \cdots & * \end{pmatrix},$$

where $p = p_1 + p_2$ and '$*$' indicates missing data. We divide the data into complete data comprising N_1 mutually independent observations in p dimensions and incomplete data comprising N_2 mutually independent observations in p_1 dimensions. Let $\boldsymbol{y}_{1j}^{(1)}$ be a $p_1 \times 1$ random vector, $\boldsymbol{y}_{2j}^{(1)}$ be a $p_2 \times 1$ random vector and $\boldsymbol{y}_j^{(1)} = (\boldsymbol{y}_{1j}^{(1)'}, \boldsymbol{y}_{2j}^{(1)'})'$, where $j = 1, 2, \ldots, N_1$.

We define the sample mean vectors as

$$\overline{\boldsymbol{y}}_1^{(1)} = \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{y}_{1j}^{(1)}, \quad \overline{\boldsymbol{y}}_2^{(1)} = \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{y}_{2j}^{(1)}, \quad \overline{\boldsymbol{y}}^{(2)} = \frac{1}{N_2}\sum_{j=1}^{N_2}\boldsymbol{y}_j^{(2)},$$

and the sample covariance matrices are denoted as

$$\boldsymbol{S}^{(1)} = \frac{1}{n_1}\sum_{j=1}^{N_1}(\boldsymbol{y}_j^{(1)} - \overline{\boldsymbol{y}}^{(1)})(\boldsymbol{y}_j^{(1)} - \overline{\boldsymbol{y}}^{(1)})' = \begin{pmatrix} \boldsymbol{S}_{11}^{(1)} & \boldsymbol{S}_{12}^{(1)} \\ \boldsymbol{S}_{21}^{(1)} & \boldsymbol{S}_{22}^{(1)} \end{pmatrix},$$
$$\boldsymbol{S}^{(2)} = \frac{1}{n_2}\sum_{j=1}^{N_2}(\boldsymbol{y}_j^{(2)} - \overline{\boldsymbol{y}}^{(2)})(\boldsymbol{y}_j^{(2)} - \overline{\boldsymbol{y}}^{(2)})',$$

where $n_i = N_i - 1$, $i = 1, 2$. The MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are denoted by $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$, respectively, and are expressed as follows (see, Kanda and Fujikoshi, 1998):

$$\widehat{\boldsymbol{\mu}} = \begin{pmatrix} \widehat{\boldsymbol{\mu}}_1 \\ \widehat{\boldsymbol{\mu}}_2 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{N}(N_1\overline{\boldsymbol{y}}_1^{(1)} + N_2\overline{\boldsymbol{y}}^{(2)}) \\ \overline{\boldsymbol{y}}_2^{(1)} - \widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\overline{\boldsymbol{y}}_1^{(1)} - \widehat{\boldsymbol{\mu}}_1) \end{pmatrix},$$

$$\widehat{\boldsymbol{\Sigma}} = \begin{pmatrix} \widehat{\boldsymbol{\Sigma}}_{11} & \widehat{\boldsymbol{\Sigma}}_{12} \\ \widehat{\boldsymbol{\Sigma}}_{21} & \widehat{\boldsymbol{\Sigma}}_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{N}(\boldsymbol{W}_{11}^{(1)} + \boldsymbol{W}^{(2)}) & \widehat{\boldsymbol{\Sigma}}_{11}(\boldsymbol{W}_{11}^{(1)})^{-1}\boldsymbol{W}_{12}^{(1)} \\ \boldsymbol{W}_{21}^{(1)}(\boldsymbol{W}_{11}^{(1)})^{-1}\widehat{\boldsymbol{\Sigma}}_{11} & \dfrac{1}{N_1}\boldsymbol{W}_{22\cdot 1}^{(1)} + \widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\widehat{\boldsymbol{\Sigma}}_{12} \end{pmatrix},$$

where

$$\boldsymbol{W}^{(1)} = n_1\boldsymbol{S}^{(1)} = \begin{pmatrix} \boldsymbol{W}_{11}^{(1)} & \boldsymbol{W}_{12}^{(1)} \\ \boldsymbol{W}_{21}^{(1)} & \boldsymbol{W}_{22}^{(1)} \end{pmatrix},$$
$$\boldsymbol{W}_{22\cdot 1}^{(1)} = \boldsymbol{W}_{22}^{(1)} - \boldsymbol{W}_{21}^{(1)}(\boldsymbol{W}_{11}^{(1)})^{-1}\boldsymbol{W}_{12}^{(1)},$$
$$\boldsymbol{W}^{(2)} = n_2\boldsymbol{S}^{(2)} + \frac{N_1N_2}{N}(\overline{\boldsymbol{y}}_1^{(1)} - \overline{\boldsymbol{y}}^{(2)})(\overline{\boldsymbol{y}}_1^{(1)} - \overline{\boldsymbol{y}}^{(2)})',$$
$$N = N_1 + N_2.$$

3 Test statistic for adequacy and its null distribution with two-step monotone missing data

We consider a test of the mean structure hypothesis

$$H_0 : \boldsymbol{\mu} = \boldsymbol{X\theta} \text{ vs. } H_1 : \boldsymbol{\mu} \neq \boldsymbol{X\theta}$$

with two-step monotone missing data in growth curve model. Then, we propose the following test statistic

$$Q = (\widetilde{\boldsymbol{X}}'\widehat{\boldsymbol{\mu}})'(\widetilde{\boldsymbol{X}}'\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}})\widetilde{\boldsymbol{X}})^{-1}(\widetilde{\boldsymbol{X}}'\widehat{\boldsymbol{\mu}}),$$

which corresponds to the test statistic T^2 in the case of complete data, where

$$\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}}) = \begin{pmatrix} \frac{1}{N}\widehat{\boldsymbol{\Sigma}}_{11} & \frac{1}{N}\widehat{\boldsymbol{\Sigma}}_{12} \\ \frac{1}{N}\widehat{\boldsymbol{\Sigma}}_{21} & \widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}}_2) \end{pmatrix},$$

which is produced by Kanda and Fujikoshi (1998). As the statistic in the case of complete data essentially follows as F-distribution, we approximate the test statistic for adequacy, Q to the F distribution, that is, we can obtain the null distribution of Q by employing the method of adjusting the second degree of freedom and the expectation of Q. Because its expectation cannot be derived precisely, it is derived using an stochastic expansion. The derivation uses the stochastic expansion of the test statistic

$$T_1^2 = \left(\sqrt{n}\widehat{\boldsymbol{\mu}}\right)' \left(n\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}})\right)^{-1} \left(\sqrt{n}\widehat{\boldsymbol{\mu}}\right)$$

for the test of the mean vector $H_{01} : \boldsymbol{\mu} = \mathbf{0}$ vs. $H_{11} : \boldsymbol{\mu} \neq \mathbf{0}$ with monotone missing data in Kawasaki and Seo (2016) since $\widetilde{\boldsymbol{X}}'\widehat{\boldsymbol{\mu}} = \mathbf{0}$ where $n = n_1 + n_2$. Here, the following equation is utilized for the stochastic expansion in a situation when

$$\gamma_i = \frac{n_i}{n} \to \text{positive constants},$$

as n_i tends to infinity, where $n_i = N_i - 1$, $i = 1, 2$. Let

$$\overline{\boldsymbol{y}}^{(1)} = \frac{1}{\sqrt{N_1}}\boldsymbol{z}^{(1)}, \quad \overline{\boldsymbol{y}}^{(2)} = \frac{1}{\sqrt{N_2}}\boldsymbol{z}^{(2)}, \quad \boldsymbol{S}^{(1)} = \boldsymbol{I}_p + \frac{1}{\sqrt{n_1}}\boldsymbol{V}^{(1)}, \quad \boldsymbol{S}^{(2)} = \boldsymbol{I}_{p_1} + \frac{1}{\sqrt{n_2}}\boldsymbol{V}^{(2)},$$

where

$$\overline{\boldsymbol{y}}^{(1)} = \begin{pmatrix} \overline{\boldsymbol{y}}_1^{(1)} \\ \overline{\boldsymbol{y}}_2^{(1)} \end{pmatrix}, \quad \boldsymbol{z}^{(1)} = \begin{pmatrix} \boldsymbol{z}_1^{(1)} \\ \boldsymbol{z}_2^{(1)} \end{pmatrix}, \quad \boldsymbol{V}^{(1)} = \begin{pmatrix} \boldsymbol{V}_{11}^{(1)} & \boldsymbol{V}_{12}^{(1)} \\ \boldsymbol{V}_{21}^{(1)} & \boldsymbol{V}_{22}^{(1)} \end{pmatrix}.$$

The test statistic Q can be expanded as follows:

$$\begin{aligned} Q = {} & \boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{s} + \frac{1}{\sqrt{n}}\left(\boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{t} - \boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{\Xi}_2\boldsymbol{\Xi}_1^{-1}\boldsymbol{s} + \boldsymbol{t}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{s}\right) \\ & + \frac{1}{n}(\boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{u} - \boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{\Xi}_2\boldsymbol{\Xi}_1^{-1}\boldsymbol{t} - \boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{\Xi}_3\boldsymbol{\Xi}_1^{-1}\boldsymbol{s} + \boldsymbol{s}(\boldsymbol{\Xi}_1^{-1}\boldsymbol{\Xi}_2)^2\boldsymbol{\Xi}_1^{-1}\boldsymbol{s} \\ & + \boldsymbol{t}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{t} - \boldsymbol{t}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{\Xi}_2\boldsymbol{\Xi}_1^{-1}\boldsymbol{s} + \boldsymbol{u}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{s}) + O_p\left(n^{-\frac{3}{2}}\right), \end{aligned}$$

where

$$\begin{aligned} \widetilde{\boldsymbol{X}} &= \begin{pmatrix} \widetilde{\boldsymbol{X}}_1 \\ \widetilde{\boldsymbol{X}}_2 \end{pmatrix}, \quad n = n_1 + n_2, \quad n_i = N_i - 1, \ i = 1, 2, \\ \boldsymbol{s} &= \sqrt{\gamma_1}\widetilde{\boldsymbol{X}}_1'\boldsymbol{z}_1^{(1)} + \sqrt{\gamma_2}\widetilde{\boldsymbol{X}}_1'\boldsymbol{z}^{(2)} + \frac{1}{\sqrt{\gamma_1}}\widetilde{\boldsymbol{X}}_2'\boldsymbol{z}_2^{(1)}, \quad \boldsymbol{t} = \frac{\sqrt{\gamma_2}}{\sqrt{\gamma_1}}\widetilde{\boldsymbol{X}}_2'\boldsymbol{V}_{21}^{(1)}\boldsymbol{z}^{(2)} - \frac{\gamma_2}{\gamma_1}\widetilde{\boldsymbol{X}}_2'\boldsymbol{V}_{21}^{(1)}\boldsymbol{z}_1^{(1)}, \\ \boldsymbol{u} &= \left(\frac{1}{2\sqrt{\gamma_1}} - 2\sqrt{\gamma_1}\right)\widetilde{\boldsymbol{X}}_1'\boldsymbol{z}_1^{(1)} + \left(\frac{1}{2\sqrt{\gamma_2}} - 2\sqrt{\gamma_2}\right)\widetilde{\boldsymbol{X}}_1'\boldsymbol{z}^{(2)} \\ &\quad + \frac{\gamma_2}{\gamma_1\sqrt{\gamma_1}}\widetilde{\boldsymbol{X}}_2'\boldsymbol{V}_{21}^{(1)}\boldsymbol{V}_{11}^{(1)}\boldsymbol{z}_1^{(1)} - \frac{\sqrt{\gamma_2}}{\gamma_1}\widetilde{\boldsymbol{X}}_2'\boldsymbol{V}_{21}^{(1)}\boldsymbol{V}_{11}^{(1)}\boldsymbol{z}^{(2)} - \frac{1}{2\gamma_1\sqrt{\gamma_1}}\widetilde{\boldsymbol{X}}_2'\boldsymbol{z}_2^{(1)}, \\ \boldsymbol{\Xi}_1 &= \widetilde{\boldsymbol{X}}_1'\widetilde{\boldsymbol{X}}_1 + \frac{1}{\gamma_1}\widetilde{\boldsymbol{X}}_2'\widetilde{\boldsymbol{X}}_2, \\ \boldsymbol{\Xi}_2 &= \sqrt{\gamma_1}\widetilde{\boldsymbol{X}}_1'\boldsymbol{V}_{11}^{(1)}\widetilde{\boldsymbol{X}}_1 + \sqrt{\gamma_2}\widetilde{\boldsymbol{X}}_1'\boldsymbol{V}^{(2)}\widetilde{\boldsymbol{X}}_1 + \frac{1}{\sqrt{\gamma_1}}\widetilde{\boldsymbol{X}}_2'\boldsymbol{V}_{21}^{(1)}\widetilde{\boldsymbol{X}}_1 + \frac{1}{\sqrt{\gamma_1}}\widetilde{\boldsymbol{X}}_1'\boldsymbol{V}_{12}^{(1)}\widetilde{\boldsymbol{X}}_2 + \frac{1}{\gamma_1\sqrt{\gamma_1}}\widetilde{\boldsymbol{X}}_2'\boldsymbol{V}_{22}^{(1)}\widetilde{\boldsymbol{X}}_2, \\ \boldsymbol{\Xi}_3 &= \gamma_1\gamma_2\widetilde{\boldsymbol{X}}_1'\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_1^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)\left(\frac{1}{\sqrt{\gamma_1}}\boldsymbol{z}_1^{(1)} - \frac{1}{\sqrt{\gamma_2}}\boldsymbol{z}^{(2)}\right)'\widetilde{\boldsymbol{X}}_1 - 4\widetilde{\boldsymbol{X}}_1'\widetilde{\boldsymbol{X}}_1 \\ &\quad + \frac{\sqrt{\gamma_2}}{\sqrt{\gamma_1}}\widetilde{\boldsymbol{X}}_1'\boldsymbol{V}^{(2)}\boldsymbol{V}_{12}^{(1)}\widetilde{\boldsymbol{X}}_2 - \frac{\gamma_2}{\gamma_1}\widetilde{\boldsymbol{X}}_1'\boldsymbol{V}_{11}^{(1)}\boldsymbol{V}_{12}^{(1)}\widetilde{\boldsymbol{X}}_2 + \frac{\sqrt{\gamma_2}}{\sqrt{\gamma_1}}\widetilde{\boldsymbol{X}}_2'\boldsymbol{V}_{21}^{(1)}\boldsymbol{V}^{(2)}\widetilde{\boldsymbol{X}}_1 \\ &\quad - \frac{\gamma_2}{\gamma_1}\widetilde{\boldsymbol{X}}_2'\boldsymbol{V}_{21}^{(1)}\boldsymbol{V}_{11}^{(1)}\widetilde{\boldsymbol{X}}_1 + \frac{p_1\gamma_2 - 2}{\gamma_1^2}\widetilde{\boldsymbol{X}}_2'\widetilde{\boldsymbol{X}}_2 - \frac{\gamma_2}{\gamma_1^2}\widetilde{\boldsymbol{X}}_2'\boldsymbol{V}_{21}^{(1)}\boldsymbol{V}_{12}^{(1)}\widetilde{\boldsymbol{X}}_2. \end{aligned}$$

We note that some aspects of the results reported by Kawasaki and Seo (2016) require revision. In this study, we adopted the updated results to derive the expansion of the test statistic Q.

Next, we consider the expectation of $\boldsymbol{V}^{(1)}, \boldsymbol{V}^{(2)}, \boldsymbol{z}_1^{(1)}, \boldsymbol{z}_2^{(1)}$ and $\boldsymbol{z}^{(2)}$ for the limit term, $1/\sqrt{n}$ term, and $1/n$ term of the test statistic Q. As they are independent of each other, we obtain the expectation of $\boldsymbol{z}_1^{(1)}, \boldsymbol{z}_2^{(1)}$ and $\boldsymbol{z}^{(2)}$ after calculating the expectation of $\boldsymbol{V}^{(1)}$ and $\boldsymbol{V}^{(2)}$. For the expectation of the limit term, since $\boldsymbol{z}_1^{(1)}, \boldsymbol{z}_2^{(1)}$ and $\boldsymbol{z}^{(2)}$ are mutually independent and each follows a multivariate standard normal distribution, only the second-order term of the expectation should be considered.

First, the expectation of the limit term is

$$\mathrm{E}\left[\boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{s}\right] = p - q.$$

For the expectation of the $1/\sqrt{n}$ term, the expectations are zero because $\boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{t}$, $\boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{\Xi}_2\boldsymbol{\Xi}_1^{-1}\boldsymbol{s}$ and $\boldsymbol{t}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{s}$ can be expressed as function of first order moments of $\boldsymbol{V}^{(1)}$ and $\boldsymbol{V}^{(2)}$, respectively. Therefore, the expectation of the $1/\sqrt{n}$ term is

$$\mathrm{E}\big[\boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{t} - \boldsymbol{s}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{\Xi}_2\boldsymbol{\Xi}_1^{-1}\boldsymbol{s} + \boldsymbol{t}'\boldsymbol{\Xi}_1^{-1}\boldsymbol{s}\big] = 0.$$

Although the $1/n$ term is numerous and complex, it can be calculated, yielding the following expectation for the test statistic:

$$\mathrm{E}[Q] = p - q + \frac{1}{N}c + O\left(N^{-\frac{3}{2}}\right),$$

where

$$\begin{aligned}
c = {} & 3\mathrm{tr}\boldsymbol{\Omega}_1^2 + \mathrm{tr}\boldsymbol{\Omega}_1^3 - 2\mathrm{tr}\boldsymbol{\Omega}_1 + \left(\frac{3\gamma_1+2}{\gamma_1^2}\right)\mathrm{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}_2' + \frac{p_1\gamma_2 - 1}{\gamma_1^2}\mathrm{tr}\boldsymbol{\Omega}_3 + \frac{2}{\gamma_1^3}\mathrm{tr}\boldsymbol{\Omega}_3^2 + \frac{1}{\gamma_1^4}\mathrm{tr}\boldsymbol{\Omega}_3^3 \\
& + \frac{3}{\gamma_1}\mathrm{tr}\boldsymbol{\Omega}_1\boldsymbol{\Omega}_2\boldsymbol{\Omega}_2' + \frac{2\gamma_1+1}{\gamma_1^3}\mathrm{tr}\boldsymbol{\Omega}_2'\boldsymbol{\Omega}_2\boldsymbol{\Omega}_3 + \left(\mathrm{tr}\boldsymbol{\Omega}_1^2 + \frac{2}{\gamma_1}\mathrm{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}_2' + \frac{1}{\gamma_1^2}\mathrm{tr}\boldsymbol{\Omega}_3^2\right)\cdot\mathrm{tr}\boldsymbol{\Omega}_1 \\
& + \left(\frac{1}{\gamma_1}\mathrm{tr}\boldsymbol{\Omega}_1^2 + \frac{\gamma_1+1}{\gamma_1^3}\mathrm{tr}\boldsymbol{\Omega}_2\boldsymbol{\Omega}_2' + \frac{1}{\gamma_1^4}\mathrm{tr}\boldsymbol{\Omega}_3^2\right)\cdot\mathrm{tr}\boldsymbol{\Omega}_3, \\
\boldsymbol{\Omega}_1 = \widetilde{\boldsymbol{X}}_1\boldsymbol{\Xi}_1^{-1}\widetilde{\boldsymbol{X}}_1', \quad & \boldsymbol{\Omega}_2 = \widetilde{\boldsymbol{X}}_1\boldsymbol{\Xi}_1^{-1}\widetilde{\boldsymbol{X}}_2', \quad \boldsymbol{\Omega}_3 = \widetilde{\boldsymbol{X}}_2\boldsymbol{\Xi}_1^{-1}\widetilde{\boldsymbol{X}}_2'.
\end{aligned}$$

The test statistic Q is assumed to be

$$Q \sim \frac{N(p-q)}{f}\cdot F_{p-q,\,f}$$

by corresponding to the case of complete data, where f is an unknown parameter. Considering the expectation of the F-distribution, we note that

$$\mathrm{E}\left[Q\right] = \frac{N(p-q)}{f-2}.$$

Therefore, the unknown parameter f is

$$f = \frac{N(N+2)(p-q)+2c}{N(p-q)+c}.$$

We propose the approximate upper 100α percentiles of the test statistic Q:

$$\frac{N(p-q)}{f} \cdot F_{p-q,\ f}(\alpha),$$

where $F_{p-q,\ f}(\alpha)$ is the upper 100α percentiles of the F-distribution with degree of freedom $p-q$ and f.

4 Simulation studies

To investigate the accuracy of the approximations, we computed the upper percentiles of the test statistic Q using Monte Carlo simulation (1,000,000 runs). The simulation results for the upper percentiles and empirical type I error of Q and their approximations in the case of two-step monotone missing data. Tables 1 and 2 present the simulated upper 100α percentiles of the test statistic Q with two-step monotone missing data. In particular, we provide q_{sim} for the following cases ($\alpha = 0.05$):

$$\text{Table 1 } : (p_1, p_2, q) = (2,2,2), n_1 = 20,30,40,50,100,150,200,400,500,800,1000, n_2 = 10,20,50,$$

$$\text{Table 2 } : (p_1, p_2, q) = (2,2,2), n_1 = 20,30,40,50,100,150,200,400,500,800,1000, n_2 = \frac{n_1}{2}, n_1, 2n_1,$$

$$\text{Table 3 } : (p_1, p_2, q) = (3,3,3), n_1 = 20,30,40,50,100,150,200,400,500,800,1000, n_2 = 10,20,50,$$

$$\text{Table 4 } : (p_1, p_2, q) = (3,3,3), n_1 = 20,30,40,50,100,150,200,400,500,800,1000, n_2 = \frac{n_1}{2}, n_1, 2n_1,$$

where q_{sim} is the upper 100α percentiles of Q obtained by simulation. The tables show q_{app}, an approximation of the upper 100α percentiles by F-approximation, and $\chi^2_{p-q}(\alpha)$, the upper 100α percentiles of the chi-squared distribution. In addition, the tables show type I error α_{app} by approximation and type I error α_{χ^2} of chi-squared distribution, the expectation of Q by simulation M_{sim}, and the expectation of Q by asymptotic expansion M_{AE}, where $\alpha_{\text{app}} = \Pr(Q > F_{p-q,f}(\alpha))$, $\alpha_{\chi^2} = \Pr(Q > \chi^2_{p-q}(\alpha))$. In this simulation, we define

$$\boldsymbol{X} = \begin{pmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} \text{ and } \widetilde{\boldsymbol{X}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{20}} \\ -\frac{1}{2} & -\frac{3}{\sqrt{20}} \\ -\frac{1}{2} & \frac{3}{\sqrt{20}} \\ \frac{1}{2} & -\frac{1}{\sqrt{20}} \end{pmatrix}$$

for $q = 2$ (Tables 1 and 2), and

$$\boldsymbol{X} = \begin{pmatrix} 1 & -5 & 5 \\ 1 & -3 & -1 \\ 1 & -1 & -4 \\ 1 & 1 & -4 \\ 1 & 3 & -1 \\ 1 & 5 & 5 \end{pmatrix} \text{ and } \widetilde{\boldsymbol{X}} = \begin{pmatrix} -\frac{\sqrt{5}}{6} & \frac{1}{2\sqrt{7}} & -\frac{1}{6\sqrt{7}} \\ \frac{7}{6\sqrt{5}} & -\frac{3}{2\sqrt{7}} & \frac{5}{6\sqrt{7}} \\ \frac{2}{3\sqrt{5}} & \frac{1}{\sqrt{7}} & -\frac{5}{3\sqrt{7}} \\ -\frac{2}{3\sqrt{5}} & \frac{1}{\sqrt{7}} & \frac{5}{3\sqrt{7}} \\ -\frac{7}{6\sqrt{5}} & -\frac{3}{2\sqrt{7}} & -\frac{5}{6\sqrt{7}} \\ \frac{\sqrt{5}}{6} & \frac{1}{2\sqrt{7}} & \frac{1}{6\sqrt{7}} \end{pmatrix}$$

for $q = 3$ (Tables 3 and 4).

As shown in Tables 1-2, as the sample size n_1 increases, the values of q_{app} obtained from the proposed approximation become closer to the values of q_{sim}. Even when the sample size is small, the values of q_{sim} provide a better approximation than the values of $\chi^2_{p-q}(\alpha)$. The values of α_{app} approaches $\alpha(= 0.05)$, rather than the values of α_{χ^2}, as the sample size increases. Similarly for the expectation, as the sample size n_1 increases, the values of M_{AE} become closer to the values of M_{sim}. Similar trends were observed in Tables 3 and 4 as in Tables 1 and 2.

Table 1: Upper 100α percentiles, type I error and expectations of Q when $(p_1, p_2, q) = (2, 2, 2)$

n_1	n_2	q_{sim}	q_{app}	α_{app}	α_{χ^2}	M_{sim}	M_{AE}
10	10	10.00	8.45	0.072	0.135	2.67	3.07
20	10	7.59	7.29	0.056	0.089	2.35	2.43
30	10	7.02	6.88	0.053	0.075	2.24	2.27
40	10	6.76	6.68	0.052	0.069	2.18	2.20
50	10	6.60	6.55	0.051	0.065	2.15	2.16
100	10	6.29	6.28	0.050	0.058	2.08	2.08
150	10	6.18	6.19	0.050	0.055	2.05	2.05
200	10	6.14	6.14	0.050	0.054	2.04	2.04
400	10	6.06	6.07	0.050	0.052	2.02	2.02
500	10	6.05	6.05	0.050	0.051	2.02	2.02
800	10	6.03	6.03	0.050	0.051	2.01	2.01
1000	10	6.02	6.02	0.050	0.051	2.01	2.01
10	20	9.89	8.30	0.073	0.134	2.66	3.06
20	20	7.50	7.18	0.056	0.086	2.33	2.41
30	20	6.98	6.82	0.053	0.074	2.23	2.26
40	20	6.70	6.63	0.052	0.068	2.18	2.19
50	20	6.58	6.51	0.051	0.064	2.14	2.15
100	20	6.29	6.27	0.051	0.057	2.07	2.08
150	20	6.19	6.18	0.050	0.055	2.05	2.05
200	20	6.14	6.13	0.050	0.054	2.04	2.04
400	20	6.08	6.06	0.050	0.052	2.02	2.02
500	20	6.05	6.05	0.050	0.051	2.02	2.02
800	20	6.02	6.03	0.050	0.051	2.01	2.01
1000	20	6.03	6.02	0.050	0.051	2.01	2.01
10	50	10.17	8.30	0.077	0.139	2.71	3.14
20	50	7.44	7.08	0.057	0.085	2.33	2.40
30	50	6.91	6.73	0.054	0.073	2.22	2.25
40	50	6.64	6.56	0.052	0.066	2.16	2.18
50	50	6.52	6.46	0.051	0.063	2.13	2.14
100	50	6.25	6.24	0.050	0.056	2.07	2.07
150	50	6.17	6.17	0.050	0.054	2.05	2.05
200	50	6.15	6.13	0.050	0.054	2.04	2.04
400	50	6.06	6.06	0.050	0.052	2.02	2.02
500	50	6.04	6.05	0.050	0.051	2.02	2.01
800	50	6.02	6.03	0.050	0.051	2.01	2.01
1000	50	6.02	6.02	0.050	0.051	2.01	2.01

Note: $\chi^2_2(0.05) = 5.99$.

Table 2: Upper 100α percentiles, type I error and expectations of Q when $(p_1, p_2, q) = (2, 2, 2)$

n_1	n_2	q_{sim}	q_{app}	α_{app}	α_{χ^2}	M_{sim}	M_{AE}
10	5	10.24	8.64	0.071	0.139	2.69	3.13
20	10	7.59	7.29	0.056	0.089	2.35	2.43
30	15	6.98	6.85	0.053	0.074	2.23	2.27
40	20	6.70	6.63	0.052	0.068	2.18	2.19
50	25	6.55	6.50	0.051	0.064	2.14	2.15
100	50	6.25	6.24	0.050	0.056	2.07	2.07
150	75	6.18	6.16	0.050	0.055	2.05	2.05
200	100	6.12	6.12	0.050	0.053	2.04	2.04
400	200	6.05	6.05	0.050	0.051	2.02	2.02
500	250	6.05	6.04	0.050	0.051	2.01	2.02
800	400	6.03	6.02	0.050	0.051	2.01	2.01
1000	500	6.01	6.02	0.050	0.050	2.01	2.01
10	10	10.00	8.45	0.072	0.135	2.67	3.07
20	20	7.50	7.18	0.056	0.086	2.33	2.41
30	30	6.90	6.78	0.053	0.073	2.22	2.25
40	40	6.66	6.58	0.052	0.067	2.17	2.18
50	50	6.52	6.46	0.051	0.063	2.13	2.14
100	100	6.24	6.22	0.050	0.056	2.07	2.07
150	150	6.15	6.15	0.050	0.054	2.04	2.05
200	200	6.11	6.11	0.050	0.053	2.03	2.03
400	400	6.05	6.05	0.050	0.052	2.02	2.02
500	500	6.05	6.04	0.050	0.051	2.01	2.02
800	800	6.02	6.02	0.050	0.051	2.01	2.01
1000	1000	6.02	6.01	0.050	0.051	2.01	2.01
10	20	9.89	8.30	0.073	0.134	2.66	3.06
20	40	7.43	7.10	0.056	0.085	2.33	2.40
30	60	6.90	6.72	0.054	0.072	2.22	2.25
40	80	6.64	6.53	0.052	0.066	2.16	2.18
50	100	6.50	6.42	0.052	0.062	2.13	2.14
100	200	6.24	6.21	0.051	0.056	2.06	2.07
150	300	6.14	6.13	0.050	0.054	2.04	2.04
200	400	6.10	6.10	0.050	0.053	2.03	2.03
400	800	6.05	6.04	0.050	0.051	2.02	2.02
500	1000	6.04	6.03	0.050	0.051	2.01	2.01
800	1600	6.02	6.02	0.050	0.051	2.01	2.01
1000	2000	6.01	6.01	0.050	0.050	2.01	2.01

Note: $\chi^2_2(0.05) = 5.99$.

Table 3: Upper 100α percentiles, type I error and expectations of Q when $(p_1, p_2, q) = (3, 3, 3)$

n_1	n_2	q_{sim}	q_{app}	α_{app}	α_{χ^2}	M_{sim}	M_{AE}
10	10	16.56	12.18	0.095	0.206	4.30	5.50
20	10	10.81	10.05	0.062	0.115	3.66	3.88
30	10	9.66	9.34	0.055	0.091	3.45	3.55
40	10	9.15	8.98	0.053	0.080	3.35	3.40
50	10	8.88	8.76	0.052	0.074	3.28	3.32
100	10	8.31	8.30	0.050	0.061	3.14	3.15
150	10	8.15	8.14	0.050	0.057	3.10	3.10
200	10	8.06	8.06	0.050	0.056	3.07	3.07
400	10	7.94	7.94	0.050	0.053	3.04	3.04
500	10	7.92	7.91	0.050	0.052	3.03	3.03
800	10	7.89	7.88	0.050	0.052	3.02	3.02
1000	10	7.88	7.86	0.050	0.051	3.01	3.02
10	20	16.69	11.96	0.100	0.208	4.33	5.56
20	20	10.65	9.89	0.062	0.112	3.65	3.85
30	20	9.56	9.23	0.056	0.089	3.44	3.52
40	20	9.06	8.90	0.053	0.078	3.33	3.38
50	20	8.82	8.70	0.052	0.073	3.27	3.30
100	20	8.30	8.28	0.050	0.061	3.14	3.15
150	20	8.12	8.13	0.050	0.057	3.10	3.09
200	20	8.07	8.06	0.050	0.056	3.07	3.08
400	20	7.95	7.94	0.050	0.053	3.04	3.04
500	20	7.92	7.91	0.050	0.052	3.03	3.03
800	20	7.89	7.88	0.050	0.052	3.02	3.02
1000	20	7.86	7.86	0.050	0.051	3.01	3.02
10	50	18.15	11.87	0.116	0.226	4.41	5.93
20	50	10.66	9.75	0.065	0.113	3.65	3.87
30	50	9.50	9.11	0.057	0.088	3.43	3.52
40	50	9.01	8.80	0.054	0.077	3.32	3.37
50	50	8.74	8.62	0.052	0.071	3.26	3.29
100	50	8.29	8.24	0.051	0.061	3.13	3.14
150	50	8.13	8.11	0.050	0.057	3.09	3.09
200	50	8.06	8.04	0.050	0.056	3.07	3.07
400	50	7.93	7.93	0.050	0.053	3.04	3.03
500	50	7.91	7.91	0.050	0.052	3.03	3.03
800	50	7.89	7.88	0.050	0.052	3.02	3.02
1000	50	7.86	7.86	0.050	0.051	3.01	3.02

Note: $\chi^2_3(0.05) = 7.81$.

Table 4: Upper 100α percentiles, type I error and expectations of Q when $(p_1, p_2, q) = (3, 3, 3)$

n_1	n_2	q_{sim}	q_{app}	α_{app}	α_{χ^2}	M_{sim}	M_{AE}
10	5	16.78	12.47	0.094	0.209	4.30	5.56
20	10	10.81	10.05	0.062	0.115	3.66	3.88
30	15	9.61	9.28	0.056	0.090	3.44	3.54
40	20	9.06	8.90	0.053	0.078	3.33	3.38
50	25	8.80	8.68	0.052	0.072	3.27	3.30
100	50	8.29	8.24	0.051	0.061	3.13	3.14
150	75	8.11	8.10	0.050	0.057	3.09	3.09
200	100	8.02	8.03	0.050	0.055	3.07	3.07
400	200	7.92	7.92	0.050	0.052	3.03	3.03
500	250	7.91	7.90	0.050	0.052	3.03	3.03
800	400	7.86	7.87	0.050	0.051	3.02	3.01
1000	500	7.87	7.86	0.050	0.051	3.01	3.02
10	10	16.56	12.18	0.095	0.206	4.30	5.50
20	20	10.65	9.89	0.062	0.112	3.65	3.85
30	30	9.49	9.17	0.056	0.088	3.43	3.51
40	40	9.03	8.82	0.054	0.077	3.32	3.37
50	50	8.74	8.62	0.052	0.071	3.26	3.29
100	100	8.24	8.21	0.051	0.060	3.13	3.13
150	150	8.10	8.08	0.051	0.056	3.09	3.10
200	200	8.02	8.01	0.050	0.055	3.06	3.06
400	400	7.92	7.91	0.050	0.052	3.03	3.03
500	500	7.91	7.89	0.050	0.052	3.03	3.03
800	800	7.87	7.86	0.050	0.051	3.02	3.02
1000	1000	7.85	7.85	0.050	0.051	3.01	3.01
10	20	16.69	11.96	0.100	0.208	4.33	5.56
20	40	10.65	9.77	0.064	0.112	3.65	3.86
30	60	9.48	9.09	0.057	0.088	3.43	3.51
40	80	8.98	8.76	0.054	0.076	3.32	3.36
50	100	8.72	8.57	0.053	0.071	3.26	3.28
100	200	8.24	8.19	0.051	0.060	3.13	3.14
150	300	8.08	8.06	0.050	0.056	3.09	3.09
200	400	8.03	8.00	0.051	0.055	3.06	3.07
400	800	7.91	7.91	0.050	0.052	3.03	3.03
500	1000	7.89	7.89	0.050	0.052	3.03	3.03
800	1600	7.84	7.86	0.050	0.051	3.02	3.01
1000	2000	7.85	7.85	0.050	0.051	3.01	3.01

Note: $\chi^2_3(0.05) = 7.81$.

5 Conclusion

We considered a test for the adequacy of a growth curve model when the data exhibit a two-step monotone missing pattern. For the mean structure hypothesis, we proposed a test statistic Q and obtained its approximate upper percentiles using an F-approximation with an adjusted second degree of freedom. The proposed approximation was more accurate than the χ^2 approximations in almost all cases.

Acknowledgments

The first and second authors' research was partly supported by a Grant-in-Aid for Early-Career Scientists (22K13961) and a Grant-in-Aid for Scientific Research (C) (21K11795), respectively. The authors thank Mr. Toya Ozaki, a graduate student at Tokyo University of Science, for his assistance with the calculations.

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