Improved Estimation of Kurtosis Parameter in Elliptical Distribution

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Abstract

In this paper, we discuss the estimation of the kurtosis parameter, which is an important parameter in elliptical distributions. In particular, we evaluate the expectation and variance of estimators of the kurtosis parameter and propose new estimators. Moreover, for the case where the covariance matrix is known, we derive the first- to third-order cumulants of the estimator and construct normalizing transformation statistics based on them to improve the normal approximation. Finally, using Monte Carlo simulations, we investigate the bias and mean squared error of these estimators, as well as the performance of the normal approximation.

Keywords: Asymptotic expansion; Elliptical distribution; Kurtosis parameter; Normalizing transformation statistic.

1 Introduction

In this paper, we discuss the estimation of the kurtosis parameter in elliptical distributions. Elliptical distributions form a class of nonnormal distributions that generalize the multivariate normal distribution. This class includes, for example, the multivariate normal, multivariate t, and contaminated normal distributions.

In multivariate statistical analysis, accurate estimation of the kurtosis parameter under elliptical distributions is an important problem. Various properties and methods related to elliptical distributions have been widely studied. In particular, Mardia (1970) introduced measures of multivariate skewness and kurtosis and derived their asymptotic distributions under multivariate normality. These measures are widely used for testing normality. Seo and Toyama (1996) derived asymptotic expansions for the expectation and variance of an estimator of the kurtosis parameter. Furthermore, Iwashita (1997) and Wakaki (1997) derived the asymptotic joint density of the sample mean vector and sample covariance matrix under elliptical distributions. In addition, Maruyama and Seo (2003) discussed general moment parameters in elliptical distributions.

In this paper, we evaluate the expectation and variance of estimators of the kurtosis parameter and propose new estimators. Moreover, under the assumption that the covariance matrix is known, we derive an asymptotic expansion of the distribution function and construct normalizing transformation statistics that improve the normal approximation, based on the method of Konishi (1981).

The remainder of this paper is organized as follows. In Section 2, we introduce elliptical distributions and discuss the estimation of the kurtosis parameter. In Section 3, we derive higher-order asymptotic expansions for the expectation and variance of the kurtosis parameter and propose new estimators. In Section 4, under the assumption that the covariance matrix is known, we derive an asymptotic expansion of the distribution function of the kurtosis estimator and construct normalizing transformation statistics that improve the normal approximation. In Section 5, Monte Carlo simulations are conducted to evaluate the bias and mean squared error (MSE) of the proposed estimators and examine the improvement of the normal approximation obtained by the normalizing transformation statistics. Finally, Section 6 discusses the results of this paper and presents concluding

remarks.

2 Elliptical Distributions and Kurtosis Parameter

We consider the same setting as Seo and Toyama (1996). Let $\boldsymbol{x}$ be a p-dimensional random vector following an elliptical distribution with parameter $\boldsymbol{\mu}$ $(p \times 1)$ and positive definite matrix $\boldsymbol{\Lambda}$ $(p \times p)$. In this case, we write $\boldsymbol{x} \sim E_p(\boldsymbol{\mu}, \boldsymbol{\Lambda})$. The probability density function is given by

$$f(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Lambda}) = c_p |\boldsymbol{\Lambda}|^{-\frac{1}{2}} g\left\{(\boldsymbol{x}-\boldsymbol{\mu})' \boldsymbol{\Lambda}^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\right\},$$

where c_p is a normalizing constant and $g(\cdot)$ is a nonnegative function. The characteristic function is given by

$$C(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{\mu})\psi(\boldsymbol{t}'\boldsymbol{\Lambda}\boldsymbol{t}),$$

where $\psi(\cdot)$ is a function. It follows that we have $\mathrm{E}[\boldsymbol{x}] = \boldsymbol{\mu}$ and $\boldsymbol{\Sigma} \equiv \mathrm{Cov}[\boldsymbol{x}] = -2\psi'(0)\boldsymbol{\Lambda}$. In addition, the elliptical distribution involves a kurtosis parameter κ, defined by

$$\kappa \equiv \frac{\psi''(0)}{\{\psi'(0)\}^2} - 1.$$

Under the multivariate normal distribution, we have $\kappa = 0$.

The multivariate kurtosis introduced by Mardia (1970) was extended to elliptical distributions by Berkane and Bentler (1987) and Anderson (1993). Under elliptical distributions, it is known that

$$\mathrm{E}\left[\{(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\}^2\right] = p(p+2)(\kappa+1).$$

Now, let $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ be a random sample from $E_p(\boldsymbol{\mu}, \boldsymbol{\Lambda})$. The multivariate sample kurtosis is defined by

$$b_{2,p} = \frac{1}{N}\sum_{i=1}^{N}\{(\boldsymbol{x}_i - \overline{\boldsymbol{x}})'\boldsymbol{S}^{-1}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})\}^2,$$

where the sample mean vector $\overline{\boldsymbol{x}}$ and sample covariance matrix $\boldsymbol{S}$ are given by

$$\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_i, \quad \boldsymbol{S} = \frac{1}{N-1}\sum_{i=1}^{N}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})'.$$

Therefore, the usual estimator of the kurtosis parameter in elliptical distributions is given by

$$\widehat{\kappa} = \frac{1}{p(p+2)}b_{2,p} - 1.$$

The multivariate kurtosis measure $b_{2,p}$ introduced by Mardia (1970) was also discussed by Anderson (1993) and is based on the matrix $\boldsymbol{S}$ defined by

$$\boldsymbol{S} = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})'.$$

However, for the sake of simplicity in the subsequent discussion, this paper defines $\boldsymbol{S}$ as the unbiased estimator of the covariance matrix. Moreover, in the discussion of estimation, we may assume, without loss of generality, that $\boldsymbol{\Sigma} = \boldsymbol{I}$ and $\boldsymbol{\mu} = \boldsymbol{0}$.

3 Proposal of Improved Estimators

3.1 Case where $\boldsymbol{\Sigma}$ is known

Let $\widehat{\kappa}_{\Sigma}$ denote an estimator of the kurtosis parameter in the case where $\boldsymbol{\Sigma}$ is known. We evaluate the expectation of $\widehat{\kappa}_{\Sigma}$ and perform bias correction by expressing it without using order notation, and then we propose improved estimators. Note that Seo and Toyama (1996) derived the asymptotic expansion only up to order N^{-1}. In this case, we can write

$$b_{2,p} = \frac{1}{N}\sum_{\alpha=1}^{N}\chi_{\alpha}^{4},$$

where $\chi_{\alpha}^{2} = (\boldsymbol{x}_{\alpha} - \overline{\boldsymbol{x}})'(\boldsymbol{x}_{\alpha} - \overline{\boldsymbol{x}})$.

In this case, $\boldsymbol{x}_{\alpha}$ and $\overline{\boldsymbol{x}}$ are dependent. To remove the dependence between $\boldsymbol{x}_{\alpha}$ and $\overline{\boldsymbol{x}}$, we use the conditional sample mean vector $\overline{\boldsymbol{x}}^{(\alpha)}$. This is defined on the subset of

$\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ obtained by deleting the observation $\boldsymbol{x}_\alpha$ at random and is given by

$$\overline{\boldsymbol{x}}^{(\alpha)} = \frac{1}{N-1} \sum_{\substack{i=1 \\ i \neq \alpha}}^{N} \boldsymbol{x}_i.$$

Then, χ^2_α can be expressed as

$$\chi^2_\alpha = \left(1 - \frac{1}{N}\right)^2 (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})'(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)}).$$

Note that $\boldsymbol{x}_\alpha$ and $\overline{\boldsymbol{x}}^{(\alpha)}$ are mutually independent.

To evaluate the expectation of $\widehat{\kappa}_\Sigma$ using a perturbation method, we set

$$\overline{\boldsymbol{x}}^{(\alpha)} = \frac{1}{\sqrt{N-1}} \boldsymbol{y}_1,$$

where $\boldsymbol{y}_1$ asymptotically follows $N_p(\boldsymbol{0}, \boldsymbol{I}_p)$. Then, χ^4_α can be expanded without using order notation as

$$\begin{aligned}
\chi^4_\alpha =& (\boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha)^2 - \frac{4\sqrt{N-1}}{N} \boldsymbol{x}'_\alpha \boldsymbol{y}_1 (\boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha) - \frac{2}{N} \left\{ \boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha (2\boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha - \boldsymbol{y}'_1 \boldsymbol{y}_1) - 2(\boldsymbol{x}'_\alpha \boldsymbol{y}_1)^2 \right\} \\
&+ \frac{4\sqrt{N-1}}{N^2} \boldsymbol{x}'_\alpha \boldsymbol{y}_1 (3\boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha - \boldsymbol{y}'_1 \boldsymbol{y}_1) \\
&+ \frac{1}{N^2} \left\{ 6\boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha (\boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha - \boldsymbol{y}'_1 \boldsymbol{y}_1) - 12(\boldsymbol{x}'_\alpha \boldsymbol{y}_1)^2 + (\boldsymbol{y}'_1 \boldsymbol{y}_1)^2 \right\} \\
&- \frac{4\sqrt{N-1}}{N^3} \boldsymbol{x}'_\alpha \boldsymbol{y}_1 (3\boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha - 2\boldsymbol{y}'_1 \boldsymbol{y}_1) \\
&- \frac{2}{N^3} \left\{ \boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha (2\boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha - 3\boldsymbol{y}'_1 \boldsymbol{y}_1) - 6(\boldsymbol{x}'_\alpha \boldsymbol{y}_1)^2 + (\boldsymbol{y}'_1 \boldsymbol{y}_1)^2 \right\} \\
&+ \frac{4\sqrt{N-1}}{N^4} \boldsymbol{x}'_\alpha \boldsymbol{y}_1 (\boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha - \boldsymbol{y}'_1 \boldsymbol{y}_1) \\
&+ \frac{1}{N^4} \left\{ \boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha (\boldsymbol{x}'_\alpha \boldsymbol{x}_\alpha - 2\boldsymbol{y}'_1 \boldsymbol{y}_1) - 4(\boldsymbol{x}'_\alpha \boldsymbol{y}_1)^2 + (\boldsymbol{y}'_1 \boldsymbol{y}_1)^2 \right\}.
\end{aligned}$$

Note that the term of order N^{-1} in this expansion coincides with the result of Seo and Toyama (1996). Moreover, because this expression is derived without invoking order notation, it can be regarded as an exact expansion, in contrast to the asymptotic expansion given by Seo and Toyama (1996), which retains terms only up to order N^{-1}. Thus,

by taking expectations with respect to $\boldsymbol{x}_\alpha$ and $\boldsymbol{y}_1$ under the elliptical distribution, the expectation of χ_α^4 is obtained as

$$\mathrm{E}\left[\chi_\alpha^4\right] = p(p+2)\left\{(\kappa+1) - \frac{2}{N}(2\kappa+1) + \frac{1}{N^2}(6\kappa+1) - \frac{4\kappa}{N^3} + \frac{\kappa}{N^4}\right\}.$$

Therefore, expressing the expectation of $\widehat{\kappa}_\Sigma$ without using order notation, we obtain

$$\mathrm{E}\left[\widehat{\kappa}_\Sigma\right] = \kappa - \frac{2}{N}(2\kappa+1) + \frac{1}{N^2}(6\kappa+1) - \frac{4\kappa}{N^3} + \frac{\kappa}{N^4}.$$

Note that this equality holds only for sufficiently large N.

Next, we derive the variance of $\widehat{\kappa}_\Sigma$ using the same approach. To obtain the variance, it suffices to evaluate the expectation of $\widehat{\kappa}_\Sigma^2$. To remove the dependence among $\boldsymbol{x}_\alpha$, $\boldsymbol{x}_\beta$, and $\overline{\boldsymbol{x}}$, we use the conditional sample mean vector $\overline{\boldsymbol{x}}^{(\alpha,\beta)}$. This is defined on the subset of $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ obtained by deleting the observations $\boldsymbol{x}_\alpha$ and $\boldsymbol{x}_\beta$ at random and is given by

$$\overline{\boldsymbol{x}}^{(\alpha,\beta)} = \frac{1}{N-2}\sum_{\substack{i=1 \\ i\neq\alpha,\beta}}^{N} \boldsymbol{x}_i.$$

Then, χ_α^2 can be expressed as

$$\chi_\alpha^2 = \left(1-\frac{1}{N}\right)^2 \left(\boldsymbol{x}_\alpha - \frac{N-2}{N-1}\overline{\boldsymbol{x}}^{(\alpha,\beta)} - \frac{1}{N-1}\boldsymbol{x}_\beta\right)' \left(\boldsymbol{x}_\alpha - \frac{N-2}{N-1}\overline{\boldsymbol{x}}^{(\alpha,\beta)} - \frac{1}{N-1}\boldsymbol{x}_\beta\right).$$

Note that $\boldsymbol{x}_\alpha$, $\boldsymbol{x}_\beta$, and $\overline{\boldsymbol{x}}^{(\alpha,\beta)}$ are mutually independent.

To evaluate the expectation of $\widehat{\kappa}_\Sigma^2$ using a perturbation method, we set

$$\overline{\boldsymbol{x}}^{(\alpha,\beta)} = \frac{1}{\sqrt{N-2}}\boldsymbol{y}_2,$$

where $\boldsymbol{y}_2$ asymptotically follows $N_p(\boldsymbol{0}, \boldsymbol{I}_p)$. In this case, χ_α^8 and $\chi_\alpha^4\chi_\beta^4$ can be expanded. Using these results, we compute $\mathrm{E}\left[\widehat{\kappa}_\Sigma^2\right]$ and obtain the variance $\sigma_{\Sigma,\mathrm{NEW}}^2$ of $Y = \sqrt{N}(\widehat{\kappa}_\Sigma - \kappa)$

without using order notation, which is given by

$$
\begin{aligned}
\sigma^2_{\boldsymbol{\Sigma},\mathrm{NEW}} = \frac{1}{p(p+2)}\Bigg[&\Big\{(p+4)(p+6)(\kappa^{(4)}+1) - p(p+2)(\kappa+1)^2\Big\} \\
&- \frac{8}{N}\Big\{(p+4)(p+6)(\kappa^{(4)}+1) - (p+4)^2(\kappa^{(3)}+1) - (p+2)^2(\kappa+1)^2 \\
&\quad + (p+2)(p+4)(\kappa+1) - (p+2)\Big\} \\
&+ \frac{4}{N^2}\Big\{7(p+4)(p+6)(\kappa^{(4)}+1) - 2(p+4)(7p+27)(\kappa^{(3)}+1) \\
&\quad - 7(p+2)(p+4)(\kappa+1)^2 + (17p^2+114p+172)(\kappa+1) - (p+2)(3p+22)\Big\} \\
&- \frac{2}{N^3}\Big\{27(p+4)(p+6)(\kappa^{(4)}+1) - 4(p+4)(18p+77)(\kappa^{(3)}+1) \\
&\quad - 2(15p^2+118p+188)(\kappa+1)^2 + (109p^2+834p+1424)(\kappa+1) \\
&\quad - 2(17p^2+136p+222)\Big\} \\
&+ \frac{2}{N^4}\Big\{34(p+4)(p+6)(\kappa^{(4)}+1) - 2(p+4)(49p+230)(\kappa^{(3)}+1) \\
&\quad - (45p^2+410p+772)(\kappa+1)^2 + (191p^2+1598p+3032)(\kappa+1) \\
&\quad - 6(13p^2+106p+192)\Big\} \\
&- \frac{2}{N^5}\Big\{22(p+4)(p+6)(\kappa^{(4)}+1) - 72(p+4)(p+5)(\kappa^{(3)}+1) \\
&\quad - 4(2p+13)(6p+19)(\kappa+1)^2 + (193p^2+1722p+3584)(\kappa+1) \\
&\quad - (91p^2+790p+1600)\Big\} \\
&+ \frac{2}{N^6}\Big\{4(p+4)(p+6)(\kappa^{(4)}+1) - 2(p+4)(11p+58)(\kappa^{(3)}+1) \\
&\quad - 2(17p^2+170p+392)(\kappa+1)^2 + (107p^2+1022p+2312)(\kappa+1) \\
&\quad - (55p^2+518p+1160)\Big\} \\
&+ \frac{1}{N^7}\Big\{2(p+4)(p+6)(\kappa^{(4)}+1) - 4(p+4)(p+6)(\kappa^{(3)}+1) \\
&\quad + (29p^2+298p+720)(\kappa+1)^2 - 2(29p^2+298p+720)(\kappa+1)
\end{aligned}
$$

$$+ (31p^2 + 318p + 768)\Big\}$$
$$+ \frac{1}{N^8}(p+4)(p+6)\Big\{-(\kappa^{(4)}+1) + 4(\kappa^{(3)}+1) - 6(\kappa+1)^2 + 6(\kappa+1) - 3\Big\}\Big],$$

where

$$\kappa^{(k)} \equiv \frac{\psi^{(k)}(0)}{\{\psi'(0)\}^k} - 1, \quad k = 3, 4.$$

Note that this result holds only for sufficiently large N. We note that the limit distribution of $Y = \sqrt{N}(\widehat{\kappa} - \kappa)$ is a normal distribution with variance given by

$$\sigma^2 = \frac{(p+4)(p+6)(\kappa^{(4)}+1)}{p(p+2)} - (\kappa+1)^2.$$

Note that $\sigma^2_{\Sigma,\mathrm{NEW}}$ denotes the exact variance of $Y = \sqrt{N}(\hat{\kappa}_\Sigma - \kappa)$, not the variance of the proposed estimator $\hat{\kappa}_{\Sigma,\mathrm{NEW}}$. Because $\hat{\kappa}_{\Sigma,\mathrm{NEW}}$ is an affine transformation of $\hat{\kappa}_\Sigma$, we have

$$\mathrm{Var}\left[\hat{\kappa}_{\Sigma,\mathrm{NEW}}\right] = \left(1 - \frac{1}{N}\right)^{-8} \frac{\sigma^2_{\Sigma,\mathrm{NEW}}}{N}.$$

Furthermore, by using $\mathrm{E}\left[\widehat{\kappa}_{\boldsymbol{\Sigma}}\right]$, we propose new estimators. The estimators $\widehat{\kappa}^*_{\boldsymbol{\Sigma}}$ and $\widehat{\kappa}^{**}_{\boldsymbol{\Sigma}}$ proposed by Seo and Toyama (1996) are given by

$$\widehat{\kappa}^*_{\boldsymbol{\Sigma}} = \left(1 + \frac{4}{N}\right)\widehat{\kappa}_{\boldsymbol{\Sigma}} + \frac{2}{N}, \quad \widehat{\kappa}^{**}_{\boldsymbol{\Sigma}} = \left(1 - \frac{4}{N}\right)^{-1}\left(\widehat{\kappa}_{\boldsymbol{\Sigma}} + 1 - \frac{2}{N}\right) - 1.$$

$\mathrm{E}\left[\widehat{\kappa}_{\boldsymbol{\Sigma}}\right]$ can be rewritten as

$$\begin{aligned}
\mathrm{E}\left[\widehat{\kappa}_{\boldsymbol{\Sigma}}\right] &= \kappa - \frac{2}{N}(2\kappa+1) + \frac{1}{N^2}(6\kappa+1) - \frac{4\kappa}{N^3} + \frac{\kappa}{N^4} \\
&= \left(1 - \frac{4}{N} + \frac{6}{N^2} - \frac{4}{N^3} + \frac{1}{N^4}\right)\kappa + \left(-\frac{2}{N} + \frac{1}{N^2}\right) \\
&= \left(1 - \frac{1}{N}\right)^4 \kappa + \left(-\frac{2}{N} + \frac{1}{N^2}\right).
\end{aligned}$$

From the above arguments, we propose an estimator of κ, denoted by $\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{NEW}}$, which is

bias-corrected up to the highest-order term, as

$$\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{NEW}} = \left(1 - \frac{1}{N}\right)^{-4} \left(\widehat{\kappa}_{\mathbf{\Sigma}} + \frac{2}{N} - \frac{1}{N^2}\right).$$

The variance of $\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{NEW}}$ coincides with that of $\widehat{\kappa}_{\mathbf{\Sigma}}$ up to order N^{-1}. Here, the estimators $\widehat{\kappa}^*_{\mathbf{\Sigma}}$ and $\widehat{\kappa}^{**}_{\mathbf{\Sigma}}$ proposed by Seo and Toyama (1996) satisfy

$$\mathrm{E}[\widehat{\kappa}^*_{\mathbf{\Sigma}}] = \kappa + O(N^{-2}), \quad \mathrm{E}[\widehat{\kappa}^{**}_{\mathbf{\Sigma}}] = \kappa + O(N^{-2}),$$

whereas for the proposed estimator $\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{NEW}}$, we have

$$\mathrm{E}[\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{NEW}}] = \kappa.$$

This equality holds only for sufficiently large N, and $\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{NEW}}$ is therefore an asymptotically unbiased estimator. Although the bias of $\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{NEW}}$ is corrected up to the highest-order term, its variance becomes larger than that of $\widehat{\kappa}_{\mathbf{\Sigma}}$. Hence, to suppress the increase in variance while correcting the bias, we propose the following estimator $\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{M}}$:

$$\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{M}} = \left(1 + \frac{6}{N^2} + \frac{1}{N^4}\right)^{-1} \left(\widehat{\kappa}_{\mathbf{\Sigma}} + \frac{2}{N} - \frac{1}{N^2}\right).$$

$\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{M}}$ is a bias-corrected estimator constructed so that the variance does not exceed that of $\widehat{\kappa}_{\mathbf{\Sigma}}$.

3.2 Case where $\mathbf{\Sigma}$ is unknown

Now, we consider the case where the covariance matrix $\mathbf{\Sigma}$ is unknown. We evaluate the expectation of $\widehat{\kappa}$ and perform bias correction by deriving its asymptotic expansion up to order N^{-2} and then propose improved estimators. Note that Seo and Toyama (1996) derived the asymptotic expansion only up to order N^{-1}. In this case, we can write

$$b_{2,p} = \frac{1}{N}\sum_{\alpha=1}^{N} T_\alpha^4,$$

where $T_\alpha^2 = (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}})' \boldsymbol{S}^{-1} (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}})$.

In this case, $\boldsymbol{x}_\alpha$, $\overline{\boldsymbol{x}}$, and $\boldsymbol{S}$ are dependent. To remove the dependence between $\boldsymbol{x}_\alpha$ and $\overline{\boldsymbol{x}}$ and between $\boldsymbol{x}_\alpha$ and $\boldsymbol{S}$, we use the conditional sample mean vector $\overline{\boldsymbol{x}}^{(\alpha)}$ and sample covariance matrix $\boldsymbol{S}^{(\alpha)}$. These are defined on the subset of $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ obtained by deleting the observation $\boldsymbol{x}_\alpha$ at random and are given by

$$\overline{\boldsymbol{x}}^{(\alpha)} = \frac{1}{N-1} \sum_{\substack{i=1 \\ i \neq \alpha}}^{N} \boldsymbol{x}_i,$$

$$\boldsymbol{S}^{(\alpha)} = \frac{1}{N-2} \sum_{\substack{i=1 \\ i \neq \alpha}}^{N} (\boldsymbol{x}_i - \overline{\boldsymbol{x}}^{(\alpha)})(\boldsymbol{x}_i - \overline{\boldsymbol{x}}^{(\alpha)})'.$$

Then, expressing $\boldsymbol{S}$ in terms of $\boldsymbol{S}^{(\alpha)}$, we have

$$\boldsymbol{S} = \left(1 - \frac{1}{N-1}\right) \boldsymbol{S}^{(\alpha)} + \frac{1}{N} (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})'.$$

Moreover, the inverse matrix $\boldsymbol{S}^{-1}$ is given by

$$\boldsymbol{S}^{-1} = \frac{N-1}{N-2} \boldsymbol{S}^{(\alpha)-1} - \frac{\frac{(N-1)^2}{N(N-2)^2} \boldsymbol{S}^{(\alpha)-1} (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})' \boldsymbol{S}^{(\alpha)-1}}{1 + \frac{N-1}{N(N-2)} (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})' \boldsymbol{S}^{(\alpha)-1} (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})}.$$

Therefore, T_α^2 can be written as

$$\begin{aligned}
T_\alpha^2 &= (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}})' \boldsymbol{S}^{-1} (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}) \\
&= \left(1 - \frac{1}{N}\right)^2 (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})' \boldsymbol{S}^{-1} (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)}) \\
&= \left(1 - \frac{1}{N}\right)^2 \left(\widetilde{T}_\alpha^2 - \frac{\frac{1}{N} \widetilde{T}_\alpha^4}{1 + \frac{1}{N} \widetilde{T}_\alpha^2} \right) \\
&= \left(1 - \frac{1}{N}\right)^2 \left(\frac{\widetilde{T}_\alpha^2}{1 + \frac{1}{N} \widetilde{T}_\alpha^2} \right),
\end{aligned}$$

where

$$\widetilde{T}_\alpha^2 = \left(1 + \frac{1}{N-2}\right) (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})' \boldsymbol{S}^{(\alpha)-1} (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)}).$$

To evaluate the expectation of $\widehat{\kappa}$ using a perturbation method, we set

$$\begin{aligned}\overline{\boldsymbol{x}}^{(\alpha)} &= \frac{1}{\sqrt{N-1}}\boldsymbol{y},\\ \boldsymbol{S}^{(\alpha)} &= \boldsymbol{I} + \frac{1}{\sqrt{N-1}}\boldsymbol{V}.\end{aligned}$$

Then, $\widetilde{T}_\alpha^2$ can be expanded up to order N^{-2} as

$$\begin{aligned}\widetilde{T}_\alpha^2 &= \frac{N-1}{N-2}(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})'\boldsymbol{S}^{(\alpha)-1}(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})\\ &= \frac{N-1}{N-2}\left(\boldsymbol{x}_\alpha - \frac{1}{\sqrt{N-1}}\boldsymbol{y}\right)'\left(\boldsymbol{I} + \frac{1}{\sqrt{N-1}}\boldsymbol{V}\right)^{-1}\left(\boldsymbol{x}_\alpha - \frac{1}{\sqrt{N-1}}\boldsymbol{y}\right)\\ &= \boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha - \frac{1}{\sqrt{N}}Q_1 + \frac{1}{N}Q_2 - \frac{1}{N\sqrt{N}}Q_3 + \frac{1}{N^2}Q_4 + O_p(N^{-\frac{5}{2}}),\end{aligned}$$

where

$$\begin{aligned}Q_1 &= 2\boldsymbol{x}_\alpha'\boldsymbol{y} + \boldsymbol{x}_\alpha'\boldsymbol{V}\boldsymbol{x}_\alpha,\\ Q_2 &= \boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha + 2\boldsymbol{x}_\alpha'\boldsymbol{V}\boldsymbol{y} + \boldsymbol{x}_\alpha'\boldsymbol{V}^2\boldsymbol{x}_\alpha + \boldsymbol{y}'\boldsymbol{y},\\ Q_3 &= 3\boldsymbol{x}_\alpha'\boldsymbol{y} + \frac{3}{2}\boldsymbol{x}_\alpha'\boldsymbol{V}\boldsymbol{x}_\alpha + \boldsymbol{y}'\boldsymbol{V}\boldsymbol{y} + 2\boldsymbol{x}_\alpha'\boldsymbol{V}^2\boldsymbol{y} + \boldsymbol{x}_\alpha'\boldsymbol{V}^3\boldsymbol{x}_\alpha,\\ Q_4 &= 2\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha + 4\boldsymbol{x}_\alpha'\boldsymbol{V}\boldsymbol{y} + 2\boldsymbol{x}_\alpha'\boldsymbol{V}^2\boldsymbol{x}_\alpha + 2\boldsymbol{x}_\alpha'\boldsymbol{V}^3\boldsymbol{y} + \boldsymbol{x}_\alpha'\boldsymbol{V}^4\boldsymbol{x}_\alpha + 2\boldsymbol{y}'\boldsymbol{y} + \boldsymbol{y}'\boldsymbol{V}^2\boldsymbol{y}.\end{aligned}$$

Expanding T_α^4, we obtain

$$\begin{aligned}T_\alpha^4 = (\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha)^2 &- \frac{2}{\sqrt{N}}\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha Q_1\\ &+ \frac{1}{N}\left\{Q_1^2 + 2\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha Q_2 - 4(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha)^2 - 2(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha)^3\right\}\\ &- \frac{2}{N\sqrt{N}}\left\{Q_1Q_2 - 4\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha Q_1 - 3(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha)^2Q_1 + \boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha Q_3\right\}\\ &+ \frac{1}{N^2}\big\{2Q_1Q_3 - 4Q_1^2 - 6\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha Q_1^2 + Q_2^2 - 8\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha Q_2 - 6(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha)^2Q_2\\ &\quad + 2\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha Q_4 + 6(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha)^2 + 8(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha)^3 + 3(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha)^4\big\} + O_p(N^{-\frac{5}{2}}).\end{aligned}$$

Note that the coefficients of the terms of order $N^{-1/2}$ and N^{-1} in the expansions of $\widetilde{T}_\alpha^2$

and T_α^4 coincide with those given by Seo and Toyama (1996); the terms of order N^{-1} and higher in $\widetilde{T}_\alpha^2$ (i.e., Q_3, Q_4) are newly derived in this paper. Note that under elliptical distributions, $\boldsymbol{V}$ and $\boldsymbol{y}$ are dependent. Therefore, to obtain the expectation of T_α^4, we make use of the result on the joint density of $\boldsymbol{V}$ and $\boldsymbol{y}$ given by Wakaki (1997).

From the above arguments, the expectation of $\widehat{\kappa}$ is obtained as

$$\mathrm{E}[\widehat{\kappa}] = \kappa - \frac{c}{N} + \frac{d}{N^2} + O(N^{-3}),$$

where

$$\begin{aligned}
c &= 2(p+4)(\kappa^{(3)}+1) - (2p+7)(\kappa+1)^2 + 5(\kappa+1) - 2, \\
d &= 3(p+4)(p+6)(\kappa^{(4)}+1) + 14(p+4)(\kappa^{(3)}+1) - 8(p+4)(p+5)(\kappa^{(3)}+1)(\kappa+1) \\
&\quad + (p+4)(5p+22)(\kappa+1)^3 - 7(2p+7)(\kappa+1)^2 - (2p+1)(\kappa+1) + 2p + 1.
\end{aligned}$$

Furthermore, by using $\mathrm{E}[\widehat{\kappa}]$, we propose new estimators. Seo and Toyama (1996) proposed the estimators $\widehat{\kappa}^*$ and $\widehat{\kappa}^{**}$. The estimator $\widehat{\kappa}^*$ is given by

$$\widehat{\kappa}^* = \widehat{\kappa} + \frac{\widehat{c}}{N},$$

where

$$\widehat{c} = 2(p+4)(\widehat{\kappa}^{(3)}+1) - (2p+7)(\widehat{\kappa}+1)^2 + 5(\widehat{\kappa}+1) - 2.$$

The estimator $\widehat{\kappa}^{**}$ is given by

$$\widehat{\kappa}^{**} = \left(1 - \frac{\widehat{c}_1}{N}\right)^{-1} \left(\widehat{\kappa} + 1 + \frac{\widehat{c}_2}{N}\right) - 1,$$

where

$$\begin{aligned}
\widehat{c}_1 &= -(2p+7)(\widehat{\kappa}+1) + 5, \\
\widehat{c}_2 &= 2(p+4)(\widehat{\kappa}^{(3)}+1) - 2.
\end{aligned}$$

Using the result of $\mathrm{E}[\widehat{\kappa}]$, we can naturally derive the following estimator $\widehat{\kappa}^{\dagger}$ by replacing c and d with their corresponding estimators:

$$\widehat{\kappa}^{\dagger} = \widehat{\kappa} + \frac{\widehat{c}}{N} - \frac{\widehat{d}}{N^2},$$

where

$$\begin{aligned}
\widehat{c} &= 2(p+4)(\widehat{\kappa}^{(3)}+1) - (2p+7)(\widehat{\kappa}+1)^2 + 5(\widehat{\kappa}+1) - 2, \\
\widehat{d} &= 3(p+4)(p+6)(\widehat{\kappa}^{(4)}+1) + 14(p+4)(\widehat{\kappa}^{(3)}+1) - 8(p+4)(p+5)(\widehat{\kappa}^{(3)}+1)(\widehat{\kappa}+1) \\
&\quad + (p+4)(5p+22)(\widehat{\kappa}+1)^3 - 7(2p+7)(\widehat{\kappa}+1)^2 - (2p+1)(\widehat{\kappa}+1) + 2p + 1.
\end{aligned}$$

However, the estimator $\widehat{\kappa}^{\dagger}$ does not take into account the bias of $\widehat{c}$. Consequently, its bias can be larger than that of $\widehat{\kappa}^{*}$. Therefore, we improve the bias correction by deriving the order N^{-1} term of $\mathrm{E}[\widehat{c}]$. As $\mathrm{E}[\widehat{c}]$ can be expressed as

$$\mathrm{E}[\widehat{c}] = 2(p+4)\mathrm{E}\left[\widehat{\kappa}^{(3)}\right] - (2p+7)\mathrm{E}\left[\widehat{\kappa}^{2}\right] - (4p+9)\mathrm{E}[\widehat{\kappa}] + 4,$$

it suffices to evaluate $\mathrm{E}\left[\widehat{\kappa}^{(3)}\right]$, $\mathrm{E}\left[\widehat{\kappa}^{2}\right]$, and $\mathrm{E}\left[\widehat{\kappa}\right]$ up to order N^{-1}. The expectation $\mathrm{E}\left[\widehat{\kappa}\right]$ has already been derived, and $\mathrm{E}\left[\widehat{\kappa}^{2}\right]$ can be obtained easily from $\mathrm{E}\left[\widehat{\kappa}^{2}\right] = \left\{\mathrm{E}[\widehat{\kappa}]\right\}^2 + \sigma^2/N$.

Therefore, the N^{-1} term of $\mathrm{E}[\widehat{c}]$ is obtained, and we propose the following estimator $\widehat{\kappa}_{\mathrm{NEW}}$, which corrects the bias of $\widehat{\kappa}$ up to order N^{-2}:

$$\widehat{\kappa}_{\mathrm{NEW}} = \widehat{\kappa} + \frac{\widehat{c}}{N} - \frac{\widehat{d}^{*}}{N^2},$$

where

$$\begin{aligned}
\widehat{c} &= 2(p+4)(\widehat{\kappa}^{(3)}+1) - (2p+7)(\widehat{\kappa}+1)^2 + 5(\widehat{\kappa}+1) - 2, \\
\widehat{d}^{*} &= -\frac{1}{p(p+2)}(p+4)(p+6)(3p^2+8p+7)(\widehat{\kappa}^{(4)}+1) - 14(p+4)(\widehat{\kappa}^{(3)}+1) \\
&\quad + \frac{2}{p}(p+2)(p+4)(3p+7)(\widehat{\kappa}^{(3)}+1)(\widehat{\kappa}+1) - \frac{1}{p}(p+4)(3p^2+10p+14)(\widehat{\kappa}+1)^3
\end{aligned}$$

$$+ 9(2p+7)(\widehat{\kappa}+1)^2 - 2(2p+15)(\widehat{\kappa}+1) + 2p + 11.$$

The variance of $\widehat{\kappa}_{\text{NEW}}$ is identical to that of $\widehat{\kappa}$ up to order N^{-1}. The estimators $\widehat{\kappa}^*$ and $\widehat{\kappa}^{**}$ proposed by Seo and Toyama (1996) satisfy

$$\mathrm{E}[\widehat{\kappa}^*] = \kappa + O(N^{-2}), \quad \mathrm{E}[\widehat{\kappa}^{**}] = \kappa + O(N^{-2}),$$

whereas for the proposed estimator $\widehat{\kappa}_{\text{NEW}}$, we have

$$\mathrm{E}[\widehat{\kappa}_{\text{NEW}}] = \kappa + O(N^{-3}).$$

Although the bias of $\widehat{\kappa}_{\text{NEW}}$ is corrected up to a higher order, its variance becomes larger than that of $\widehat{\kappa}$. Therefore, to suppress the increase in variance while correcting the bias, we propose the following estimator $\widehat{\kappa}_{\text{M}}$:

$$\widehat{\kappa}_{\text{M}} = \left\{1 + \frac{1}{N}(4p+9) + \frac{1}{N^2}(7p^2+24p+5)\right\}^{-1}\left(\widehat{\kappa} + \frac{4}{N} - \frac{7}{N^2}\right).$$

$\widehat{\kappa}_{\text{M}}$ is a bias-corrected estimator constructed so that the variance does not exceed that of $\widehat{\kappa}$.

Moreover, the limiting term of the variance σ^2 of $Y = \sqrt{N}(\widehat{\kappa} - \kappa)$ given by Seo and Toyama (1996) is

$$\begin{aligned}\sigma^2 = {} & \frac{1}{p(p+2)}(p+4)(p+6)(\kappa^{(4)}+1) - \frac{4}{p}(p+4)(\kappa^{(3)}+1)(\kappa+1) \\ & + \frac{4}{p}(p+2)(\kappa+1)^3 - (\kappa+1)^2.\end{aligned}$$

4 Derivation of the Normalizing Transformation Statistic

In this section, under the assumption that the covariance matrix $\mathbf{\Sigma}$ is known, we derive an asymptotic expansion of the distribution function of Y/σ, where $Y = \sqrt{N}(\widehat{\kappa}_{\Sigma} - \kappa)$, and construct normalizing transformation statistics that improve the normal approximation.

The case where $\mathbf{\Sigma}$ is unknown is briefly discussed at the end of this section, as the corresponding derivation becomes considerably more complicated.

In this case, the distribution function of Y can be expanded as

$$\Pr\left[\frac{Y}{\sigma} \leq y\right] = \Phi(y) - \frac{1}{\sqrt{N}}\left\{\frac{a_1}{\sigma}\Phi^{(1)}(y) + \frac{a_3}{\sigma^3}\Phi^{(3)}(y)\right\} + O(N^{-1}),$$

where $\Phi(y)$ denotes the distribution function of the standard normal distribution $N(0,1)$, and $\Phi^{(j)}(y)$ denotes the jth derivative of $\Phi(y)$ $(j = 1, 3)$. Moreover, a_1, σ^2, and a_3 are the coefficients of the leading terms of the first-, second-, and third-order cumulants of Y, respectively, given by

$$\begin{aligned} K_1(Y) &= \frac{a_1}{\sqrt{N}} + O(N^{-\frac{3}{2}}), \\ K_2(Y) &= \sigma^2 + O(N^{-1}), \\ K_3(Y) &= \frac{6a_3}{\sqrt{N}} + O(N^{-\frac{3}{2}}). \end{aligned}$$

Furthermore, to eliminate the term of order $N^{-\frac{1}{2}}$ in the asymptotic expansion of the distribution function of Y, the function $g(\widehat{\kappa}_{\mathbf{\Sigma}})$ is chosen so as to satisfy the differential equation

$$\frac{a_3}{\sigma^3} + \frac{\sigma g''(\widehat{\kappa}_{\mathbf{\Sigma}})}{2g'(\widehat{\kappa}_{\mathbf{\Sigma}})} = 0 \quad (g'(\kappa) \neq 0).$$

We then obtain

$$\Pr\left[\frac{\sqrt{N}\{g(\widehat{\kappa}_{\mathbf{\Sigma}}) - g(\kappa) - c/N\}}{g'(\kappa)\sigma} \leq y\right] = \Phi(y) + O(N^{-1}),$$

where

$$c = \left(a_1 - \frac{a_3}{\sigma^2}\right) g'(\kappa).$$

The explicit forms of a_1, σ^2, a_3, $g(\widehat{\kappa}_{\mathbf{\Sigma}})$, and c are given by

$$a_1 = -2(2\kappa + 1),$$

$$
\begin{aligned}
\sigma^2 &= \frac{1}{p(p+2)}(p+4)(p+6)(\kappa^{(4)}+1) - (\kappa+1)^2, \\
a_3 &= \frac{1}{6p^2(p+2)^2}(p+4)(p+6)(p+8)(p+10)(\kappa^{(6)}+1) \\
&\quad - \frac{1}{2p(p+2)}(p+4)(p+6)(\kappa^{(4)}+1)(\kappa+1) + \frac{1}{3}(\kappa+1)^3, \\
g(\widehat{\kappa}_{\Sigma}) &= -\frac{\sigma^4}{2a_3}\exp\left[-\frac{2a_3}{\sigma^4}\widehat{\kappa}_{\Sigma}\right], \\
c &= \left(a_1 - \frac{a_3}{\sigma^2}\right)\exp\left[-\frac{2a_3}{\sigma^4}\kappa\right],
\end{aligned}
$$

where

$$
\kappa^{(k)} \equiv \frac{\psi^{(k)}(0)}{\{\psi'(0)\}^k} - 1, \quad k = 4, 6.
$$

Note that a_1 and σ^2 coincide with the corresponding quantities derived by Seo and Toyama (1996), whereas a_3, which involves the third cumulant, is newly derived in this paper. Therefore, the following normalizing transformation statistic Z_{NT} is obtained:

$$
Z_{NT} = \sqrt{N}\left[\frac{\sigma^3}{2a_3}\left\{1 - \exp\left[\frac{2a_3}{\sigma^4}(\kappa - \widehat{\kappa}_{\Sigma})\right]\right\} - \frac{1}{N\sigma}\left(a_1 - \frac{a_3}{\sigma^2}\right)\right].
$$

Let $\widehat{\kappa}_{\Sigma}$ be the multivariate sample kurtosis based on a random sample of size N drawn from $E_p(\boldsymbol{\mu}, \boldsymbol{\Lambda})$. Then, Z_{NT} converges in distribution to the standard normal distribution $N(0,1)$.

We also consider the same transformation for the estimator $\widehat{\kappa}_{\Sigma,\mathrm{NEW}}$ proposed in Section 3. In this case, only the leading term of the first cumulant is modified. The explicit forms of $a_{1,\mathrm{NEW}}$, $g(\widehat{\kappa}_{\Sigma,\mathrm{NEW}})$, and c_{NEW} are given by

$$
\begin{aligned}
a_{1,\mathrm{NEW}} &= 0, \\
g(\widehat{\kappa}_{\Sigma,\mathrm{NEW}}) &= -\frac{\sigma^4}{2a_3}\exp\left[-\frac{2a_3}{\sigma^4}\widehat{\kappa}_{\Sigma,\mathrm{NEW}}\right], \\
c_{\mathrm{NEW}} &= -\frac{a_3}{\sigma^2}\exp\left[-\frac{2a_3}{\sigma^4}\kappa\right].
\end{aligned}
$$

Therefore, the following normalizing transformation statistic $Z_{NT,\mathrm{NEW}}$ is obtained:

$$Z_{NT,\mathrm{NEW}} = \sqrt{N}\left[\frac{\sigma^3}{2a_3}\left\{1-\exp\left[\frac{2a_3}{\sigma^4}\left(\kappa-\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{NEW}}\right)\right]\right\}+\frac{a_3}{N\sigma^3}\right].$$

Finally, when the covariance matrix $\boldsymbol{\Sigma}$ is unknown, the derivation of normalizing transformation statistics results in highly complicated expressions, making the required calculations extremely difficult. Nevertheless, applying the same functional form of $Z_{NT,\mathrm{NEW}}$ by substituting $\widehat{\kappa}_{\mathrm{NEW}}$ for $\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{NEW}}$ may still improve the normal approximation even when $\boldsymbol{\Sigma}$ is unknown, although a rigorous derivation of the exact coefficients σ^2 and a_3 in this case remains intractable.

5 Simulation Studies

5.1 Bias and MSE of the Estimators

In this subsection, we investigate the bias and MSE of the estimators proposed in Section 3 by means of Monte Carlo simulations under several selected parameter settings. The simulations are conducted for elliptical distributions, and the distributions considered are the

(a) multivariate normal distribution ($\kappa = 0$),

(b) multivariate t distribution with 13 degrees of freedom ($\kappa = 0.22$), and

(c) contaminated normal distribution with $\omega = 0.1$ and $\tau = 3$ ($\kappa = 1.78$).

The simulations are performed for both the cases where $\boldsymbol{\Sigma}$ is known and unknown, with dimensions $p = 2, 3$, and 5 and sample sizes $N = 10, 20, 50, 100$, and 200. For the three elliptical populations listed above, the theoretical values of the moment parameters ($m = 2, 3, 4, 6$) are given in Table 1.

Moreover, without loss of generality, we may set $\boldsymbol{\Sigma} = \boldsymbol{I}$. Based on 100,000 Monte Carlo simulations for each parameter setting, Tables 2, 3, and 4 report the bias and MSE of $\widehat{\kappa}_{\boldsymbol{\Sigma}}$, $\widehat{\kappa}^*_{\boldsymbol{\Sigma}}$, $\widehat{\kappa}^{**}_{\boldsymbol{\Sigma}}$, $\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{NEW}}$, and $\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{M}}$ in the case where $\boldsymbol{\Sigma}$ is known. Similarly, in the case where $\boldsymbol{\Sigma}$ is unknown, Tables 5, 6, and 7 report the bias and MSE of $\widehat{\kappa}$, $\widehat{\kappa}^*$, $\widehat{\kappa}^{**}$, $\widehat{\kappa}^{\dagger}$, $\widehat{\kappa}_{\mathrm{NEW}}$, and $\widehat{\kappa}_{\mathrm{M}}$.

Table 1: Theoretical values of the moment parameters $\kappa^{(m)}$

	κ	$\kappa^{(3)}$	$\kappa^{(4)}$	$\kappa^{(6)}$
(a)	0	0	0	0
(b)	0.22	0.92	3.22	169.42
(c)	1.78	11.65	61.58	1561.52

When $\boldsymbol{\Sigma}$ is known, it is observed that $\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{NEW}}$ achieves the smallest bias under all considered settings. Moreover, because $\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{NEW}}$ also yields a smaller MSE than $\widehat{\kappa}^{**}_{\boldsymbol{\Sigma}}$, it can be regarded as an improvement over $\widehat{\kappa}^{**}_{\boldsymbol{\Sigma}}$ proposed by Seo and Toyama (1996). Meanwhile, $\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{M}}$ consistently attains the smallest MSE across all settings.

When the covariance matrix $\boldsymbol{\Sigma}$ is unknown, it is found that, for non-normal populations, $\widehat{\kappa}_{\mathrm{NEW}}$ has the smallest bias when $N \geq 50$. This result is due to the bias correction up to order N^{-2} achieved by the asymptotic expansion. Furthermore, for the multivariate normal population, $\widehat{\kappa}_{\mathrm{M}}$ minimizes both the bias and MSE simultaneously. This is because $\widehat{\kappa}_{\mathrm{M}}$ is derived from $\mathrm{E}[\widehat{\kappa}]$ under the assumption $\kappa^{(3)} = \kappa^{(4)} = \kappa^{(6)} = 0$ in order to prevent an increase in variance. Consequently, $\widehat{\kappa}_{\mathrm{M}}$ is an effective estimator only when the kurtosis parameter κ is small.

Table 2: Multivariate normal ($\boldsymbol{\Sigma}$: known, $\kappa = 0$)

		$\widehat{\kappa}_{\boldsymbol{\Sigma}}$		$\widehat{\kappa}^{*}_{\boldsymbol{\Sigma}}$		$\widehat{\kappa}^{**}_{\boldsymbol{\Sigma}}$		$\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{NEW}}$		$\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{M}}$	
p	N	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	10	0.191	0.392	0.068	0.701	0.014	0.987	**0.002**	0.826	0.010	**0.314**
	20	0.099	0.221	0.018	0.305	0.002	0.330	**0.000**	0.319	**0.000**	**0.205**
2	50	0.040	0.095	0.003	0.109	**0.000**	0.110	**0.000**	0.110	**0.000**	**0.093**
	100	0.020	0.049	0.001	0.052	**0.000**	0.053	**0.000**	0.053	**0.000**	**0.048**
	200	0.010	**0.025**	**0.000**	0.026	**0.000**	0.026	**0.000**	0.026	**0.000**	**0.025**
	10	0.191	0.265	0.067	0.452	0.015	0.635	**0.001**	0.531	0.011	**0.202**
	20	0.098	0.145	0.018	0.196	0.002	0.212	**0.001**	0.204	**0.001**	**0.132**
3	50	0.040	0.062	0.003	0.070	**0.000**	0.071	**0.000**	0.071	**0.000**	**0.060**
	100	0.020	**0.031**	0.001	0.034	**0.000**	0.034	**0.000**	0.034	**0.000**	**0.031**
	200	0.010	**0.016**	**0.000**	0.016	**0.000**	0.016	**0.000**	0.016	**0.000**	**0.016**
	10	0.191	0.168	0.067	0.262	0.016	0.365	**0.001**	0.305	0.011	**0.116**
	20	0.098	0.087	0.017	0.112	0.003	0.121	**0.001**	0.117	**0.001**	**0.075**
5	50	0.040	0.036	0.003	0.040	**0.000**	0.041	**0.000**	0.040	**0.000**	**0.034**
	100	0.020	**0.018**	0.001	0.019	**0.000**	0.019	**0.000**	0.019	**0.000**	**0.018**
	200	0.010	**0.009**	**0.000**	**0.009**	**0.000**	**0.009**	**0.000**	**0.009**	**0.000**	**0.009**

Table 3: Multivariate t with 13 degrees of freedom ($\boldsymbol{\Sigma}$: known, $\kappa = 0.22$)

		$\widehat{\kappa}_{\boldsymbol{\Sigma}}$		$\widehat{\kappa}^{*}_{\boldsymbol{\Sigma}}$		$\widehat{\kappa}^{**}_{\boldsymbol{\Sigma}}$		$\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{NEW}}$		$\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{M}}$	
p	N	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	10	0.267	1.401	0.085	2.614	0.036	3.695	**0.001**	3.089	0.074	**1.180**
	20	0.139	0.928	0.022	1.309	0.007	1.420	**0.000**	1.370	0.043	**0.884**
2	50	0.058	0.420	0.005	0.486	**0.001**	0.493	**0.001**	0.490	0.018	**0.415**
	100	0.028	0.227	0.001	0.244	0.001	0.245	**0.000**	0.245	0.009	**0.226**
	200	0.014	**0.115**	**0.000**	0.119	**0.000**	0.120	**0.000**	0.120	0.004	**0.115**
	10	0.268	1.043	0.086	1.911	0.035	2.700	**0.002**	2.257	0.075	**0.864**
	20	0.140	0.645	0.023	0.901	0.006	0.977	**0.001**	0.942	0.044	**0.609**
3	50	0.057	0.286	0.004	0.330	0.001	0.334	**0.000**	0.332	0.018	**0.282**
	100	0.028	0.157	**0.000**	0.169	0.001	0.169	**0.000**	0.169	0.008	**0.156**
	200	0.014	**0.079**	**0.000**	0.082	**0.000**	0.082	**0.000**	0.082	0.005	**0.079**
	10	0.269	0.639	0.087	1.118	0.034	1.575	**0.003**	1.316	0.076	**0.506**
	20	0.139	0.403	0.023	0.553	0.006	0.600	**0.001**	0.579	0.043	**0.374**
5	50	0.058	0.187	0.005	0.214	**0.001**	0.217	**0.001**	0.216	0.019	**0.183**
	100	0.028	0.102	0.001	0.109	0.001	0.110	**0.000**	0.109	0.008	**0.101**
	200	0.014	**0.050**	**0.000**	0.052	**0.000**	0.052	**0.000**	0.052	0.004	**0.050**

Table 4: Contaminated normal with $\omega = 0.1$ and $\tau = 3$ ($\boldsymbol{\Sigma}$: known, $\kappa = 1.78$)

		$\widehat{\kappa}_{\boldsymbol{\Sigma}}$		$\widehat{\kappa}^{*}_{\boldsymbol{\Sigma}}$		$\widehat{\kappa}^{**}_{\boldsymbol{\Sigma}}$		$\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{NEW}}$		$\widehat{\kappa}_{\boldsymbol{\Sigma},\mathrm{M}}$	
p	N	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	10	0.801	18.742	0.211	35.519	0.183	50.310	**0.001**	42.046	0.670	**16.438**
	20	0.426	13.196	0.056	18.743	0.037	20.336	**0.001**	19.616	0.349	**12.748**
2	50	0.176	6.450	0.008	7.487	0.006	7.584	**0.001**	7.545	0.141	**6.408**
	100	0.091	3.438	0.004	3.710	**0.000**	3.722	**0.000**	3.717	0.073	**3.431**
	200	0.046	1.775	0.001	1.845	**0.000**	1.846	**0.000**	1.846	0.036	**1.774**
	10	0.800	13.231	0.209	24.721	0.185	35.007	**0.002**	29.248	0.669	**11.570**
	20	0.428	9.217	0.058	13.012	0.034	14.116	**0.001**	13.616	0.351	**8.887**
3	50	0.177	4.477	0.009	5.185	0.005	5.252	**0.001**	5.225	0.142	**4.444**
	100	0.088	2.392	0.003	2.579	0.004	2.588	**0.001**	2.584	0.069	**2.387**
	200	0.046	1.234	0.002	1.282	**0.001**	1.283	**0.001**	1.282	0.037	**1.233**
	10	0.796	9.096	0.204	16.626	0.191	23.540	**0.008**	19.656	0.665	**7.918**
	20	0.422	6.213	0.051	8.693	0.042	9.431	**0.006**	9.097	0.345	**5.974**
5	50	0.177	2.979	0.009	3.438	0.006	3.483	**0.001**	3.465	0.141	**2.953**
	100	0.089	1.592	0.002	1.714	0.002	1.719	**0.001**	1.717	0.070	**1.588**
	200	0.046	0.820	0.002	0.851	**0.001**	0.852	**0.001**	0.851	0.037	**0.819**

Table 5: Multivariate normal ($\boldsymbol{\Sigma}$: unknown, $\kappa = 0$)

		$\widehat{\kappa}$		$\widehat{\kappa}^{*}$		$\widehat{\kappa}^{**}$		$\widehat{\kappa}^{\dagger}$		$\widehat{\kappa}_{\text{NEW}}$		$\widehat{\kappa}_{\text{M}}$	
p	N	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	10	0.338	0.126	0.208	0.077	0.236	0.081	0.174	0.078	0.209	0.089	**0.002**	**0.001**
	20	0.183	0.051	0.071	0.044	0.092	0.040	0.057	0.047	0.055	0.054	**0.001**	**0.004**
2	50	0.078	0.019	0.016	0.022	0.022	0.021	0.013	0.021	0.008	0.025	**0.000**	**0.007**
	100	0.040	0.010	0.004	0.011	0.006	0.011	0.004	0.011	0.001	0.012	**0.000**	**0.006**
	200	0.020	0.005	0.001	0.005	0.001	0.005	0.001	0.005	**0.000**	0.005	**0.000**	**0.004**
	10	0.338	0.119	0.206	0.058	0.243	0.069	0.174	0.051	0.219	0.070	**0.002**	**0.000**
	20	0.183	0.042	0.069	0.025	0.096	0.025	0.056	0.025	0.055	0.032	**0.000**	**0.001**
3	50	0.078	0.013	0.014	0.012	0.023	0.011	0.013	0.012	0.006	0.014	**0.000**	**0.003**
	100	0.039	0.006	0.004	0.006	0.006	0.006	0.004	0.006	0.001	0.006	**0.000**	**0.003**
	200	0.020	0.003	0.001	0.003	0.002	0.003	0.001	0.003	**0.000**	0.003	**0.000**	**0.002**
	10	0.337	0.115	0.211	0.048	0.260	0.070	0.166	0.033	0.265	0.075	**0.001**	**0.000**
	20	0.183	0.036	0.072	0.013	0.107	0.017	0.054	0.011	0.065	0.016	**0.000**	**0.000**
5	50	0.077	0.009	0.014	0.005	0.026	0.005	0.013	0.005	0.007	0.006	**0.000**	**0.001**
	100	0.039	0.003	0.004	0.003	0.007	0.003	0.004	0.003	0.001	0.003	**0.000**	**0.001**
	200	0.020	**0.001**	0.001	**0.001**	0.002	**0.001**	0.001	**0.001**	**0.000**	**0.001**	**0.000**	**0.001**

Table 6: Multivariate t with 13 degrees of freedom ($\boldsymbol{\Sigma}$: unknown, $\kappa = 0.22$)

		$\widehat{\kappa}$		$\widehat{\kappa}^{*}$		$\widehat{\kappa}^{**}$		$\widehat{\kappa}^{\dagger}$		$\widehat{\kappa}_{\text{NEW}}$		$\widehat{\kappa}_{\text{M}}$	
p	N	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	10	0.534	0.300	0.387	0.192	0.423	0.208	0.346	0.179	0.385	0.202	**0.217**	**0.048**
	20	0.337	0.144	0.191	0.109	0.224	0.105	0.170	0.113	**0.163**	0.123	0.189	**0.043**
2	50	0.167	0.069	0.062	0.087	0.077	0.078	0.060	0.086	**0.038**	0.111	0.125	**0.037**
	100	0.093	0.045	0.024	0.065	0.029	0.060	0.024	0.062	**0.011**	0.078	0.078	**0.033**
	200	0.049	0.031	0.007	0.044	0.009	0.042	0.009	0.041	**0.001**	0.050	0.045	**0.026**
	10	0.540	0.298	0.394	0.173	0.437	0.203	0.356	0.153	0.403	0.187	**0.220**	**0.049**
	20	0.346	0.134	0.198	0.077	0.236	0.083	0.180	0.073	**0.167**	0.084	0.197	**0.041**
3	50	0.175	0.052	0.066	0.052	0.084	0.047	0.069	0.048	**0.039**	0.069	0.138	**0.029**
	100	0.098	0.030	0.025	0.039	0.032	0.037	0.028	0.036	**0.011**	0.050	0.088	**0.022**
	200	0.052	0.019	0.008	0.026	0.010	0.025	0.011	0.024	**0.002**	0.030	0.052	**0.016**
	10	0.550	0.304	0.417	0.178	0.470	0.223	0.367	0.141	0.470	0.226	**0.222**	**0.049**
	20	0.361	0.135	0.219	0.061	0.266	0.080	0.198	0.054	**0.197**	0.060	0.208	**0.044**
5	50	0.188	0.044	0.077	0.028	0.100	0.028	0.082	0.025	**0.046**	0.036	0.157	**0.028**
	100	0.108	0.021	0.030	0.021	0.040	0.019	0.035	0.018	**0.013**	0.027	0.105	**0.017**
	200	0.059	0.012	0.010	0.014	0.013	0.014	0.013	0.013	**0.003**	0.018	0.063	**0.010**

Table 7: Contaminated normal with $\omega = 0.1$ and $\tau = 3$ ($\boldsymbol{\Sigma}$: unknown, $\kappa = 1.78$)

		$\widehat{\kappa}$		$\widehat{\kappa}^*$		$\widehat{\kappa}^{**}$		$\widehat{\kappa}^{\dagger}$		$\widehat{\kappa}_{\text{NEW}}$		$\widehat{\kappa}_{\text{M}}$	
p	N	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	10	2.029	4.141	1.836	3.443	1.891	3.624	1.773	3.255	1.823	3.410	**1.756**	**3.084**
	20	1.643	2.827	1.356	2.135	1.448	2.306	**1.278**	2.057	1.283	**2.013**	1.624	2.667
2	50	1.039	1.437	0.660	1.179	0.759	**1.154**	0.607	1.263	**0.529**	1.244	1.180	1.582
	100	0.634	0.814	0.283	0.853	0.347	**0.780**	0.277	0.893	**0.175**	1.011	0.774	0.899
	200	0.349	**0.462**	0.093	0.556	0.121	0.522	0.103	0.552	**0.036**	0.641	0.449	0.490
	10	2.061	4.256	1.883	3.573	1.940	3.781	1.833	3.401	1.876	3.556	**1.767**	**3.124**
	20	1.714	2.993	1.430	2.194	1.525	2.427	1.386	2.115	**1.324**	**1.972**	1.675	2.815
3	50	1.122	1.445	0.711	0.934	0.824	1.001	0.700	0.964	**0.536**	**0.904**	1.280	1.724
	100	0.698	0.717	0.308	0.558	0.385	**0.535**	0.321	0.564	**0.175**	0.652	0.863	0.900
	200	0.393	0.350	0.107	0.340	0.142	**0.321**	0.118	0.331	**0.041**	0.395	0.509	0.418
	10	2.096	4.397	1.952	3.816	2.010	4.043	1.899	3.613	1.998	3.998	**1.776**	**3.155**
	20	1.813	3.303	1.573	2.520	1.663	2.796	1.554	2.473	**1.474**	**2.251**	1.732	3.002
5	50	1.270	1.688	0.851	0.915	0.979	1.094	0.880	0.978	**0.635**	**0.698**	1.433	2.080
	100	0.819	0.778	0.384	0.375	0.484	0.417	0.421	0.398	**0.214**	**0.364**	1.022	1.106
	200	0.471	0.316	0.136	0.186	0.187	0.184	0.162	**0.183**	**0.050**	0.208	0.628	0.465

5.2 Performance of the normal approximation

In this subsection, we investigate the performance of the normal approximation for the normalizing transformation statistics by means of Monte Carlo simulations. In particular, we compare the proposed statistic $Z_{NT,\text{NEW}}$ with the standardized statistics Z, Z_{NEW}, and Z^*_{NEW}. Z denotes the statistic obtained by standardizing $\widehat{\kappa}_{\boldsymbol{\Sigma}}$ using the limiting values of its expectation and variance. Similarly, Z_{NEW} is the standardized statistic based on $\widehat{\kappa}_{\boldsymbol{\Sigma},\text{NEW}}$ using the limiting expectation and variance. Moreover, Z^*_{NEW} is constructed by standardizing $\widehat{\kappa}_{\boldsymbol{\Sigma},\text{NEW}}$ using the expectation and variance expanded up to the highest-order terms in the asymptotic expansion.

The simulations are conducted under elliptical distributions. The target distributions considered are the

- multivariate normal distribution ($\kappa = 0$) and
- contaminated normal distribution with $\omega = 0.1$ and $\tau = 3$ ($\kappa = 1.78$).

In the case where $\boldsymbol{\Sigma}$ is known, Monte Carlo simulations with 100,000 replications are conducted for each $p = 2, 3$, and 5. Without loss of generality, we set $\boldsymbol{\Sigma} = \boldsymbol{I}$.

Tables 8 and 9 report the sample mean, variance, kurtosis, and skewness of Z, Z_{NEW}, Z^*_{NEW}, and $Z_{NT,\text{NEW}}$ for $N = 20, 50, 100, 200, 400$, and 800. In addition, Tables 10 and 11

present the upper $100\alpha\%$ quantiles and corresponding Type I error rates for $N = 100, 200, 400,$ and 800, with $\alpha = 0.05$. The Type I error rates are defined as

$$\begin{aligned}
\alpha &= \Pr\{Z > z_\alpha\}, \\
\alpha_{\text{NEW}} &= \Pr\{Z_{\text{NEW}} > z_\alpha\}, \\
\alpha^*_{\text{NEW}} &= \Pr\{Z^*_{\text{NEW}} > z_\alpha\}, \\
\alpha_{NT,\text{NEW}} &= \Pr\{Z_{NT,\text{NEW}} > z_\alpha\},
\end{aligned}$$

where z_α denotes the upper $100\alpha\%$ quantile of the standard normal distribution $N(0, 1)$.

As a result, Z^*_{NEW} exhibits good convergence of its mean and variance, even for small sample sizes. In contrast, with respect to kurtosis and skewness, $Z_{NT,\text{NEW}}$, which incorporates a normalizing transformation, shows faster convergence to the normal distribution than the other statistics. In particular, as the kurtosis parameter κ increases, such as in the case of the contaminated normal distribution, the convergence of the normal approximation becomes slower. In these situations, the effectiveness of the normalizing transformation is more pronounced. Furthermore, in terms of the upper percentiles and Type I error rates, $Z_{NT,\text{NEW}}$ also demonstrates faster convergence than the competing statistics. Overall, these results confirm that $Z_{NT,\text{NEW}}$ provides an improved normal approximation.

The four statistics compared in this subsection are defined as follows:

$$\begin{aligned}
Z &= \frac{\sqrt{N}\left(\widehat{\kappa}_{\Sigma} - \kappa\right)}{\sigma}, \\
Z_{\text{NEW}} &= \frac{\sqrt{N}\left(\widehat{\kappa}_{\Sigma,\text{NEW}} - \kappa\right)}{\sigma}, \\
Z^*_{\text{NEW}} &= \left(1 - \frac{1}{N}\right)^4 \frac{\sqrt{N}\left(\widehat{\kappa}_{\Sigma,\text{NEW}} - \kappa\right)}{\sigma_{\Sigma,\text{NEW}}},
\end{aligned}$$

where σ^2 is the limiting variance given in Section 4 and $\sigma^2_{\Sigma,\text{NEW}}$ is the exact variance derived in Section 3. The statistic $Z_{NT,\text{NEW}}$ is given in Section 4.

5.3 Performance of the normal approximation when $\boldsymbol{\Sigma}$ is unknown

In this subsection, we investigate the performance of the normal approximation when the covariance matrix $\boldsymbol{\Sigma}$ is unknown. As noted in Section 4, a rigorous derivation of the normalizing transformation statistic is intractable in this case. However, $\widehat{\kappa}$, $\widehat{\kappa}^*$, $\widehat{\kappa}^{**}$, $\widehat{\kappa}^\dagger$, and $\widehat{\kappa}_{\text{NEW}}$ share the same limiting variance σ^2, given in Section 3.2, up to order N^{-1}. Therefore, we standardize each estimator using this common σ^2 and examine the resulting normal approximation. In this subsection, we focus on the following three statistics:

$$Z = \frac{\sqrt{N}(\widehat{\kappa} - \kappa)}{\sigma},$$
$$Z_{\text{NEW}} = \frac{\sqrt{N}(\widehat{\kappa}_{\text{NEW}} - \kappa)}{\sigma},$$
$$Z_{\text{M}} = \frac{\sqrt{N}(\widehat{\kappa}_{\text{M}} - \kappa)}{\sigma}.$$

Z denotes the statistic obtained by standardizing $\widehat{\kappa}$ using σ. Similarly, Z_{NEW} and Z_{M} are the standardized statistics based on $\widehat{\kappa}_{\text{NEW}}$ and $\widehat{\kappa}_{\text{M}}$, respectively. In addition, we consider the statistic obtained by applying the functional form of $Z_{NT,\text{NEW}}$ in Section 4 to $\widehat{\kappa}_{\text{NEW}}$, given by

$$Z_{NT,\text{NEW}} = \sqrt{N}\left[\frac{\sigma_*^3}{2a_{3*}}\left\{1 - \exp\left[\frac{2a_{3*}}{\sigma_*^4}(\kappa - \widehat{\kappa}_{\text{NEW}})\right]\right\} + \frac{a_{3*}}{N\sigma_*^3}\right],$$

where σ_*^2 and a_{3*} are the quantities derived in Section 4 for the case where $\boldsymbol{\Sigma}$ is known, applied here without modification.

The simulations are conducted under the same settings as in Section 5.2: multivariate normal distribution ($\kappa = 0$) and contaminated normal distribution with $\omega = 0.1$ and $\tau = 3$ ($\kappa = 1.78$) for $p = 2, 3$, and 5. Monte Carlo simulations with 100,000 replications are conducted for each parameter setting. Tables 12 and 13 report the sample mean, variance, skewness, and kurtosis of Z, Z_{NEW}, Z_{M}, and $Z_{NT,\text{NEW}}$ for $N = 20, 50, 100, 200, 400$, and 800. Tables 14 and 15 present the upper $100\alpha\%$ quantiles and corresponding Type I error rates for $N = 100, 200, 400$, and 800, with $\alpha = 0.05$.

As a result, $Z_{NT,\text{NEW}}$ attains the smallest skewness among the four statistics in almost

all cases, and its mean remains close to zero for all N. With respect to kurtosis, $Z_{NT,\mathrm{NEW}}$ performs best in most settings, although for the contaminated normal distribution with small p and N, its kurtosis occasionally undershoots below 3, resulting in a larger deviation than that of Z or Z_{M}. However, its variance does not converge to 1 as N increases, and instead stabilizes around 0.13–0.32. This is because σ_*^2 and a_{3*} do not describe the actual variance of $\widehat{\kappa}_{\mathrm{NEW}}$ when $\mathbf{\Sigma}$ is unknown; in particular, the exact variance given in Section 3.2 contains an additional term involving $\kappa^{(3)}$ that has no counterpart in σ_*^2. Consequently, the Type I error rate of $Z_{NT,\mathrm{NEW}}$ remains close to 0 rather than the nominal level of 0.05 (Tables 14 and 15), indicating an overly conservative test.

These results suggest that the functional form of $Z_{NT,\mathrm{NEW}}$ retains some of its shape-correcting property when $\mathbf{\Sigma}$ is unknown but that this substitution alone is not sufficient for a valid normalizing transformation. A rigorous derivation of σ^2 and a_3 for the case where $\mathbf{\Sigma}$ is unknown remains an important topic for future work.

Table 8: Multivariate normal ($\kappa = 0$)

		Expectation				Variance			
p	N	Z	Z_{NEW}	Z^*_{NEW}	$Z_{NT,\text{NEW}}$	Z	Z_{NEW}	Z^*_{NEW}	$Z_{NT,\text{NEW}}$
2	20	−0.200	−0.006	**−0.005**	−0.026	0.845	1.274	**0.997**	0.987
	50	−0.126	**−0.001**	**−0.001**	−0.005	0.935	1.099	**0.998**	0.971
	100	−0.094	**−0.005**	**−0.005**	−0.006	0.964	1.045	**0.996**	0.980
	200	−0.063	**0.000**	**0.000**	**0.000**	0.986	1.026	**1.002**	0.992
	400	−0.044	**0.000**	**0.000**	**0.000**	0.988	1.008	**0.996**	0.990
	800	−0.033	**0.000**	**0.000**	**0.000**	0.996	1.006	**1.000**	0.998
3	20	−0.245	**−0.001**	**−0.001**	−0.030	0.850	1.282	**1.001**	1.072
	50	−0.157	**0.000**	**0.000**	−0.008	0.944	1.110	**1.007**	1.022
	100	−0.103	0.008	0.008	**0.006**	0.969	1.050	**1.000**	1.003
	200	−0.079	**0.000**	**0.000**	**0.000**	0.977	1.017	**0.993**	0.996
	400	−0.053	**0.003**	**0.003**	**0.003**	0.985	1.004	**0.992**	0.994
	800	−0.042	**−0.002**	**−0.002**	**−0.002**	0.992	1.001	**0.995**	0.996
5	20	−0.319	**0.004**	**0.004**	−0.025	0.854	1.287	**1.004**	1.157
	50	−0.208	**−0.001**	**−0.001**	−0.007	0.933	1.096	**0.994**	1.050
	100	−0.151	**−0.004**	**−0.004**	−0.007	0.964	1.045	**0.995**	1.023
	200	−0.104	**0.000**	**0.000**	−0.001	0.979	1.019	**0.994**	1.007
	400	−0.074	**0.000**	**0.000**	**0.000**	0.986	1.006	**0.993**	0.998
	800	−0.050	**0.002**	**0.002**	**0.002**	**1.003**	1.013	1.007	1.009

		Skewness				Kurtosis			
p	N	Z	Z_{NEW}	Z^*_{NEW}	$Z_{NT,\text{NEW}}$	Z	Z_{NEW}	Z^*_{NEW}	$Z_{NT,\text{NEW}}$
2	20	1.491	1.491	1.491	**−0.142**	7.174	7.174	7.174	**2.440**
	50	0.941	0.941	0.941	**−0.066**	4.665	4.665	4.665	**2.722**
	100	0.657	0.657	0.657	**−0.029**	3.796	3.796	3.796	**2.841**
	200	0.461	0.461	0.461	**−0.017**	3.388	3.388	3.388	**2.917**
	400	0.330	0.330	0.330	**−0.003**	3.189	3.189	3.189	**2.950**
	800	0.225	0.225	0.225	**−0.011**	3.103	3.103	3.103	**2.990**
3	20	1.160	1.160	1.160	**−0.120**	5.375	5.375	5.375	**2.586**
	50	0.741	0.741	0.741	**−0.037**	4.017	4.017	4.017	**2.801**
	100	0.521	0.521	0.521	**−0.015**	3.507	3.507	3.507	**2.910**
	200	0.357	0.357	0.357	**−0.010**	3.216	3.216	3.216	**2.950**
	400	0.251	0.251	0.251	**−0.006**	3.102	3.102	3.102	**2.965**
	800	0.193	0.193	0.193	**0.010**	3.076	3.076	3.076	**2.994**
5	20	0.855	0.855	0.855	**−0.101**	4.383	4.383	4.383	**2.741**
	50	0.528	0.528	0.528	**−0.036**	3.536	3.536	3.536	**2.898**
	100	0.369	0.369	0.369	**−0.011**	3.217	3.217	3.217	**2.914**
	200	0.268	0.268	0.268	**0.001**	3.125	3.125	3.125	**2.965**
	400	0.214	0.214	0.214	**0.024**	3.101	3.101	3.101	**2.996**
	800	0.139	0.139	0.139	**0.006**	3.029	3.029	3.029	**2.986**

Table 9: Contaminated normal with $\omega = 0.1$ and $\tau = 3$ ($\kappa = 1.78$)

		Expectation				Variance			
p	N	Z	Z_{NEW}	Z^*_{NEW}	$Z_{NT,\text{NEW}}$	Z	Z_{NEW}	Z^*_{NEW}	$Z_{NT,\text{NEW}}$
2	20	−0.098	**0.001**	**0.001**	0.367	0.704	1.061	**0.994**	0.311
	50	−0.066	**0.000**	**0.000**	0.151	0.865	1.016	**0.991**	0.514
	100	−0.050	**−0.003**	**−0.003**	0.064	0.936	1.014	**1.001**	0.667
	200	−0.036	**−0.003**	**−0.003**	0.026	0.963	1.003	**0.996**	0.788
	400	−0.025	**−0.001**	**−0.001**	0.009	0.989	1.009	**1.006**	0.877
	800	−0.018	**−0.001**	**−0.001**	0.003	0.989	**0.999**	**0.999**	0.930
3	20	−0.122	**−0.002**	**−0.002**	0.214	0.699	1.054	**0.987**	0.414
	50	−0.079	**−0.001**	**−0.001**	0.078	0.873	1.027	**1.001**	0.629
	100	−0.055	**0.001**	**0.001**	0.033	0.941	1.020	**1.007**	0.767
	200	−0.044	**−0.004**	**−0.004**	0.010	0.956	**0.995**	0.989	0.856
	400	−0.028	**0.000**	**0.000**	0.005	0.983	1.003	**1.000**	0.920
	800	−0.020	**0.000**	**0.000**	0.002	0.989	**0.999**	**0.999**	0.957
5	20	−0.146	**0.001**	**0.001**	0.099	0.708	1.067	**0.997**	0.564
	50	−0.102	**−0.006**	**−0.006**	0.026	0.867	1.019	**0.992**	0.757
	100	−0.063	**0.007**	**0.007**	0.018	0.941	1.019	**1.006**	0.861
	200	−0.049	**0.000**	**0.000**	0.004	0.973	1.013	**1.006**	0.925
	400	−0.032	**0.003**	**0.003**	0.004	0.984	1.004	**1.001**	0.960
	800	−0.022	**0.003**	**0.003**	0.004	0.983	**0.992**	**0.992**	0.971

		Skewness				Kurtosis			
p	N	Z	Z_{NEW}	Z^*_{NEW}	$Z_{NT,\text{NEW}}$	Z	Z_{NEW}	Z^*_{NEW}	$Z_{NT,\text{NEW}}$
2	20	4.121	4.121	4.121	**0.501**	33.039	33.039	33.039	**1.872**
	50	2.746	2.746	2.746	**0.165**	16.713	16.713	16.713	**1.923**
	100	1.982	1.982	1.982	**0.044**	10.581	10.581	10.581	**2.166**
	200	1.391	1.391	1.391	**−0.012**	6.787	6.787	6.787	**2.443**
	400	1.016	1.016	1.016	**−0.012**	5.177	5.177	5.177	**2.660**
	800	0.699	0.699	0.699	**−0.020**	3.991	3.991	3.991	**2.843**
3	20	3.187	3.187	3.187	**0.410**	19.832	19.832	19.832	**1.866**
	50	2.086	2.086	2.086	**0.130**	10.537	10.537	10.537	**2.053**
	100	1.473	1.473	1.473	**0.032**	6.725	6.725	6.725	**2.335**
	200	1.030	1.030	1.030	**−0.004**	4.846	4.846	4.846	**2.588**
	400	0.755	0.755	0.755	**−0.002**	4.035	4.035	4.035	**2.790**
	800	0.523	0.523	0.523	**−0.011**	3.531	3.531	3.531	**2.898**
5	20	2.339	2.339	2.339	**0.316**	11.964	11.964	11.964	**1.910**
	50	1.493	1.493	1.493	**0.101**	6.712	6.712	6.712	**2.240**
	100	1.064	1.064	1.064	**0.028**	4.961	4.961	4.961	**2.519**
	200	0.761	0.761	0.761	**0.021**	3.967	3.967	3.967	**2.729**
	400	0.524	0.524	0.524	**0.002**	3.455	3.455	3.455	**2.849**
	800	0.356	0.356	0.356	**−0.007**	3.177	3.177	3.177	**2.903**

Table 10: Multivariate normal ($\kappa = 0$)

		Upper percentile				Type I error rate			
p	N	Z	Z_{NEW}	Z^*_{NEW}	$Z_{NT,\text{NEW}}$	α	α_{NEW}	α^*_{NEW}	$\alpha_{NT,\text{NEW}}$
2	100	1.666	1.784	1.827	**1.614**	0.051	0.062	0.065	**0.047**
	200	1.680	1.757	1.778	**1.631**	0.053	0.060	0.062	**0.049**
	400	1.681	1.732	1.743	**1.641**	0.053	0.058	0.059	**0.050**
	800	1.664	1.699	1.704	**1.635**	0.052	0.055	0.056	**0.049**
3	100	1.634	1.773	1.816	**1.647**	0.049	0.062	0.066	**0.050**
	200	1.645	1.737	1.759	**1.644**	0.050	0.058	0.060	**0.050**
	400	1.647	1.710	1.720	**1.642**	0.050	0.056	0.057	**0.050**
	800	1.650	1.693	1.698	**1.644**	0.050	0.055	0.055	**0.050**
5	100	1.561	1.735	1.778	**1.657**	0.043	0.058	0.062	**0.051**
	200	1.591	1.709	1.730	**1.649**	0.045	0.056	0.058	**0.050**
	400	1.614	1.694	1.704	**1.648**	0.047	0.055	0.056	**0.050**
	800	1.635	1.690	1.695	**1.656**	0.049	0.054	0.055	**0.051**

Table 11: Contaminated normal with $\omega = 0.1$ and $\tau = 3$ ($\kappa = 1.78$)

		Upper percentile				Type I error rate			
p	N	Z	Z_{NEW}	Z^*_{NEW}	$Z_{NT,\text{NEW}}$	α	α_{NEW}	α^*_{NEW}	$\alpha_{NT,\text{NEW}}$
2	100	**1.787**	1.909	1.898	1.419	**0.058**	0.066	0.065	0.017
	200	1.813	1.884	1.878	**1.490**	**0.062**	0.067	0.067	0.030
	400	1.806	1.848	1.845	**1.552**	0.062	0.065	0.065	**0.039**
	800	1.761	1.787	1.786	**1.580**	0.060	0.062	0.062	**0.043**
3	100	**1.786**	1.918	1.906	1.486	**0.060**	0.070	0.069	0.030
	200	1.777	1.854	1.848	**1.539**	**0.059**	0.066	0.065	0.038
	400	1.780	1.826	1.823	**1.593**	0.061	0.065	0.065	**0.044**
	800	1.736	1.765	1.763	**1.606**	0.058	0.060	0.060	**0.046**
5	100	1.748	1.891	1.879	**1.558**	**0.058**	0.068	0.067	0.039
	200	1.745	1.830	1.824	**1.595**	0.058	0.065	0.065	**0.045**
	400	1.732	1.784	1.781	**1.620**	0.058	0.063	0.062	**0.047**
	800	1.708	1.741	1.740	**1.628**	0.056	0.059	0.059	**0.048**

Table 12: Multivariate normal ($\kappa = 0$, $\boldsymbol{\Sigma}$ unknown)

		Expectation				Variance			
p	N	Z	Z_{NEW}	Z_{M}	$Z_{NT,\text{NEW}}$	Z	Z_{NEW}	Z_{M}	$Z_{NT,\text{NEW}}$
2	20	−0.821	−0.248	**−0.003**	0.085	0.350	**1.018**	0.083	0.180
	50	−0.548	−0.047	**−0.002**	0.098	0.660	**1.262**	0.350	0.213
	100	−0.396	−0.014	**−0.003**	0.080	0.804	**1.154**	0.579	0.211
	200	−0.282	−0.004	**−0.002**	0.060	0.900	**1.089**	0.762	0.208
	400	−0.199	**0.000**	**0.000**	0.044	**0.956**	1.053	0.879	0.206
	800	−0.143	**−0.002**	**−0.002**	0.030	**0.978**	1.026	0.938	0.203
3	20	−1.124	−0.336	**−0.003**	0.016	0.315	**1.104**	0.055	0.174
	50	−0.747	−0.057	**0.000**	0.072	0.628	**1.309**	0.288	0.196
	100	−0.539	−0.014	**−0.001**	0.064	0.788	**1.187**	0.526	0.186
	200	−0.382	**0.000**	0.002	0.050	**0.902**	1.115	0.734	0.180
	400	−0.276	**−0.004**	**−0.004**	0.034	0.942	**1.047**	0.849	0.172
	800	−0.195	**−0.002**	**−0.002**	0.024	0.965	**1.017**	0.916	0.168
5	20	−1.716	−0.612	**−0.003**	−0.103	0.244	**1.064**	0.024	0.139
	50	−1.141	−0.100	**0.001**	0.038	0.576	**1.385**	0.199	0.164
	100	−0.823	−0.023	**−0.001**	0.045	**0.761**	**1.239**	0.437	0.150
	200	−0.587	−0.005	**−0.001**	0.036	**0.875**	**1.125**	0.659	0.139
	400	−0.420	−0.005	**−0.003**	0.025	0.923	**1.046**	0.800	0.130
	800	−0.299	**−0.004**	**−0.004**	0.018	**0.977**	1.040	0.909	0.130

		Skewness				Kurtosis			
p	N	Z	Z_{NEW}	Z_{M}	$Z_{NT,\text{NEW}}$	Z	Z_{NEW}	Z_{M}	$Z_{NT,\text{NEW}}$
2	20	1.087	1.560	1.087	**0.664**	5.045	7.149	5.045	**3.399**
	50	1.057	1.661	1.057	**0.768**	5.351	9.423	5.351	**4.122**
	100	0.838	1.112	0.838	**0.634**	4.516	5.820	4.516	**3.878**
	200	0.659	0.783	0.659	**0.497**	3.999	4.471	3.999	**3.622**
	400	0.492	0.542	0.492	**0.362**	3.607	3.770	3.607	**3.391**
	800	0.339	0.355	0.339	**0.243**	3.241	3.275	3.241	**3.133**
3	20	0.841	1.328	0.841	**0.693**	4.136	5.951	4.136	**3.535**
	50	0.840	1.363	0.840	**0.777**	4.491	7.239	4.491	**4.247**
	100	0.715	1.064	0.715	**0.631**	4.401	8.153	4.401	**4.004**
	200	0.544	0.668	0.544	**0.456**	3.803	4.421	3.803	**3.646**
	400	0.383	0.423	0.383	**0.303**	3.360	3.455	3.360	**3.268**
	800	0.279	0.292	0.279	**0.214**	3.165	3.188	3.165	**3.106**
5	20	0.619	0.935	0.619	**0.590**	3.552	4.354	3.552	**3.362**
	50	**0.624**	1.039	**0.624**	0.721	**3.736**	5.280	**3.736**	4.050
	100	**0.515**	0.748	**0.515**	0.553	**3.547**	4.376	**3.547**	3.755
	200	0.400	0.493	0.400	**0.378**	**3.361**	3.604	**3.361**	3.373
	400	0.303	0.335	0.303	**0.263**	3.205	3.264	3.205	**3.171**
	800	0.222	0.233	0.222	**0.184**	3.092	3.105	3.092	**3.064**

Table 13: Contaminated normal with $\omega = 0.1$ and $\tau = 3$ ($\kappa = 1.78$, $\mathbf{\Sigma}$ unknown)

		Expectation				Variance			
p	N	Z	Z_{NEW}	Z_{M}	$Z_{NT,\text{NEW}}$	Z	Z_{NEW}	Z_{M}	$Z_{NT,\text{NEW}}$
2	20	−0.679	−0.531	−0.670	**0.329**	0.022	**0.062**	0.005	0.043
	50	−0.678	−0.344	−0.772	**0.181**	0.151	**0.412**	0.080	0.154
	100	−0.582	−0.156	−0.711	**0.151**	0.361	**0.859**	0.260	0.238
	200	−0.457	**−0.049**	−0.583	0.129	0.576	**1.082**	0.487	0.285
	400	−0.342	**−0.011**	−0.445	0.102	0.757	**1.111**	0.696	0.310
	800	−0.254	**−0.008**	−0.333	0.073	0.861	**1.058**	0.825	0.316
3	20	−0.916	−0.708	−0.896	**0.080**	0.016	**0.062**	0.003	0.037
	50	−0.948	−0.451	−1.083	**0.045**	0.134	**0.445**	0.061	0.153
	100	−0.834	−0.208	−1.032	**0.075**	0.331	**0.897**	0.221	0.235
	200	−0.667	**−0.072**	−0.866	0.086	0.555	**1.114**	0.451	0.274
	400	−0.499	**−0.016**	−0.667	0.078	0.739	**1.123**	0.666	0.286
	800	−0.360	**0.000**	−0.490	0.062	0.858	**1.077**	0.814	0.283
5	20	−1.347	−1.095	−1.286	**−0.235**	0.008	**0.043**	0.001	0.020
	50	−1.490	−0.742	−1.683	**−0.152**	0.105	**0.409**	0.036	0.122
	100	−1.358	−0.351	−1.695	**−0.026**	0.293	**0.876**	0.168	0.199
	200	−1.107	−0.120	−1.473	**0.038**	0.522	**1.139**	0.393	0.236
	400	−0.841	**−0.032**	−1.161	0.052	0.712	**1.141**	0.617	0.237
	800	−0.613	**−0.004**	−0.865	0.046	0.840	**1.088**	0.782	0.229

		Skewness				Kurtosis			
p	N	Z	Z_{NEW}	Z_{M}	$Z_{NT,\text{NEW}}$	Z	Z_{NEW}	Z_{M}	$Z_{NT,\text{NEW}}$
2	20	0.943	0.757	0.943	**0.406**	3.710	**2.844**	3.710	2.191
	50	0.918	1.100	0.918	**0.248**	4.105	4.442	4.105	**2.409**
	100	0.971	1.477	0.971	**0.242**	4.760	6.966	4.760	**2.782**
	200	0.898	1.454	0.898	**0.296**	4.670	7.498	4.670	**3.098**
	400	0.809	1.166	0.809	**0.328**	4.533	6.282	4.533	**3.302**
	800	0.631	0.791	0.631	**0.279**	3.978	4.609	3.978	**3.281**
3	20	0.693	0.443	0.693	**0.210**	**3.277**	2.323	**3.277**	2.066
	50	0.584	0.778	0.584	**0.134**	**3.345**	3.678	**3.345**	2.542
	100	0.580	1.099	0.580	**0.223**	3.608	5.340	3.608	**2.966**
	200	0.621	1.129	0.621	**0.316**	3.923	5.958	3.923	**3.303**
	400	0.557	0.872	0.557	**0.310**	3.808	5.059	3.808	**3.372**
	800	0.478	0.633	0.478	**0.268**	3.669	4.298	3.669	**3.339**
5	20	0.523	0.127	0.523	**−0.012**	**3.148**	2.276	**3.148**	2.204
	50	0.275	0.277	0.275	**−0.113**	**2.964**	2.918	**2.964**	2.733
	100	0.237	0.613	0.237	**0.093**	3.166	3.883	3.166	**3.131**
	200	0.306	0.734	0.306	**0.260**	**3.325**	4.485	**3.325**	3.354
	400	0.313	0.573	0.313	**0.252**	3.350	3.988	3.350	**3.348**
	800	0.283	0.404	0.283	**0.198**	3.292	3.536	3.292	**3.242**

Table 14: Multivariate normal ($\kappa = 0$, $\mathbf{\Sigma}$ unknown)

		Upper percentile				Type I error rate			
p	N	Z	Z_{NEW}	Z_{M}	$Z_{NT,\text{NEW}}$	α	α_{NEW}	α_{M}	$\alpha_{NT,\text{NEW}}$
2	100	1.223	1.932	**1.372**	0.897	0.026	0.071	**0.030**	0.004
	200	1.405	1.858	**1.550**	0.858	0.034	0.066	**0.043**	0.003
	400	1.510	1.799	**1.639**	0.825	0.040	0.063	**0.050**	0.002
	800	1.568	1.750	**1.674**	0.798	0.044	0.060	**0.052**	0.001
3	100	1.056	**1.958**	1.302	0.833	0.018	**0.074**	0.025	0.002
	200	1.289	1.869	**1.509**	0.789	0.027	0.069	**0.039**	0.002
	400	1.417	1.785	**1.604**	0.749	0.033	0.062	**0.046**	0.001
	800	1.490	1.732	**1.640**	0.722	0.038	0.058	**0.050**	0.000
5	100	0.719	**1.961**	1.167	0.727	0.008	**0.075**	0.015	0.001
	200	1.043	**1.854**	1.414	0.681	0.016	**0.069**	0.030	0.000
	400	1.237	1.765	**1.539**	0.643	0.022	0.061	**0.041**	0.000
	800	1.387	1.738	**1.623**	0.628	0.030	0.059	**0.048**	0.000

Table 15: Contaminated normal with $\omega = 0.1$ and $\tau = 3$ ($\kappa = 1.78$, $\mathbf{\Sigma}$ unknown)

		Upper percentile				Type I error rate			
p	N	Z	Z_{NEW}	Z_{M}	$Z_{NT,\text{NEW}}$	α	α_{NEW}	α_{M}	$\alpha_{NT,\text{NEW}}$
2	100	0.521	**1.588**	0.225	1.005	0.004	**0.047**	0.001	0.001
	200	0.915	**1.840**	0.680	1.056	0.014	**0.062**	0.008	0.006
	400	1.224	**1.891**	1.056	1.065	0.025	**0.067**	0.018	0.007
	800	1.380	**1.814**	1.267	1.028	0.033	**0.064**	0.027	0.006
3	100	0.188	**1.549**	−0.197	0.915	0.001	**0.044**	0.000	0.001
	200	0.659	**1.829**	0.331	0.989	0.006	**0.063**	0.002	0.005
	400	1.007	**1.862**	0.763	0.990	0.015	**0.067**	0.009	0.005
	800	1.262	**1.827**	1.090	0.969	0.025	**0.065**	0.018	0.004
5	100	−0.438	**1.326**	−0.998	0.727	0.000	**0.030**	0.000	0.000
	200	0.134	**1.792**	−0.396	0.874	0.001	**0.061**	0.000	0.002
	400	0.613	**1.849**	0.193	0.882	0.005	**0.066**	0.001	0.002
	800	0.954	**1.793**	0.647	0.850	0.012	**0.064**	0.005	0.002

6 Conclusions

In this paper, we investigated the estimation of the kurtosis parameter under elliptical distributions. When the covariance matrix $\mathbf{\Sigma}$ is known, we evaluated the expectation of $\widehat{\kappa}_{\mathbf{\Sigma}}$ without using order notation and proposed two new estimators, $\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{NEW}}$ and $\widehat{\kappa}_{\mathbf{\Sigma},\mathrm{M}}$. When $\mathbf{\Sigma}$ is unknown, we derived the expectation of $\widehat{\kappa}$ up to order N^{-2} and proposed the improved estimators $\widehat{\kappa}^{\dagger}$, $\widehat{\kappa}_{\mathrm{NEW}}$, and $\widehat{\kappa}_{\mathrm{M}}$. The effectiveness of these estimators was demonstrated through extensive simulation studies.

Furthermore, in the case where $\mathbf{\Sigma}$ is known, we derived the first three cumulants of $Y = \sqrt{N}(\widehat{\kappa}_{\mathbf{\Sigma}} - \kappa)$ and proposed normalizing transformation statistics to improve the normal approximation. The improvement in the normal approximation was also confirmed by simulation results. Meanwhile, when $\mathbf{\Sigma}$ is unknown, the derivation of normalizing transformation statistics results in highly complicated expressions, making the required calculations extremely difficult.

As a topic for future research, it would be of interest to evaluate the variance of $\widehat{\kappa}$ up to order N^{-2} and to develop higher-order standardized statistics based on such expansions.

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