Low dimensional topology and complex analysis (2)

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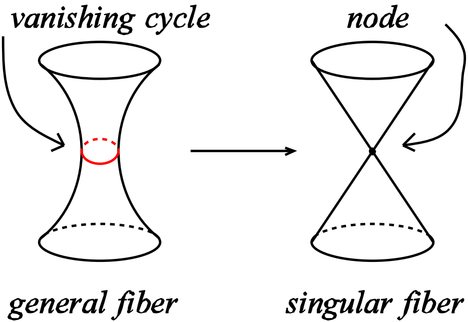
contents

Structure o f a Lefschtz type singular fiber

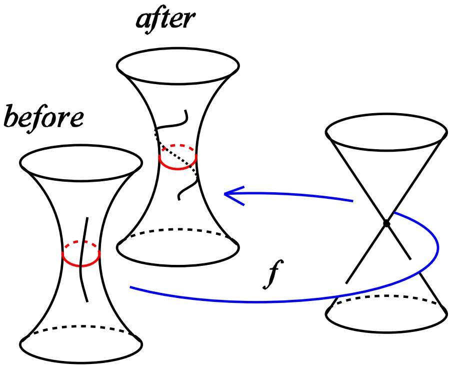
Node (ordinary double point)

$$(z_1, z_2) \mapsto z_1^2 + z_2^2 = (z_1 + \sqrt{-1}z_2)(z_1 - \sqrt{-1}z_2)$$



Dehn twist

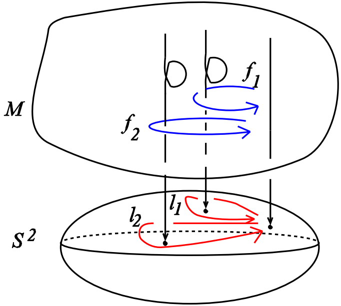
Monodromy around a Lefschetz type singular fiber is a (right handed = negative) Dehn twist about the vanishing cycle:



Application (a relation in Γ_g produces a 4-manifold)

A relation of Dehn twists gives a Lefschetz fibration over S^2:

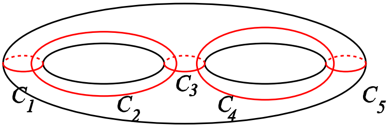
$C_1, C_2, \ldots, C_r$ **simple closed curves in Σ_g.**
If $D(C_1)D(C_2)\cdots D(C_r) = 1$ in Γ_g, then we get a Lefschetz fibration:



$f_1 = D(C_1),\ f_2 = D(C_2),\ \cdots,$ **NB:** $l_1 l_2 \cdots l_r \simeq 0$ **in** S^2.

Examples in $g = 2$

Now let $C_1, C_2, \ldots, C_5$ denote standard curves on Σ_2:



Denote $\zeta_i = D(C_i), i = 1, 2, \ldots, 5$ (negative Dehn twists.)

Well known relations;

(A) $(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5^2\zeta_4\zeta_3\zeta_2\zeta_1)^2 = 1$ gives $\mathbb{C}P^2\#13\overline{\mathbb{C}P^2}$,

(B) $(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5)^6 = 1$ gives $K3\#2\overline{\mathbb{C}P^2}$.

(C) $(\zeta_1\zeta_2\zeta_3\zeta_4)^{10} = 1$ and $(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5^2\zeta_4\zeta_3\zeta_2\zeta_1)^4 = 1$

The two relations in (C) give homeomorphic but non-diffeomorphic 4-manifolds. (T. Fuller, 1996)

Siebert-Tian Conjecture, 1990's

If a Lefschetz fibration of genus 2 over S^2 has only non-separating vanishing cycles, then it is a fiber connected sum of copies of the above three examples (A), (B), (C). (A higher genus version exists.)

Unsolved (until now).

Is Kamada's Chart theory useful?

Close relationship with symplectic 4-manifolds (1)

Definition

A 4-manifold with a 2-form ω satisfying

- $\omega^2 \neq 0$
- $d\omega = 0$

is called a symplectic 4-manifold.

Example: An algebraic surface is a symplectic 4-manifold.

Close relationship with symplectic 4-manifolds (2)

In 1990's,

S. K. Donaldson proved : A symplectic 4-manifold admits a Lefschetz pencil.

and conversely

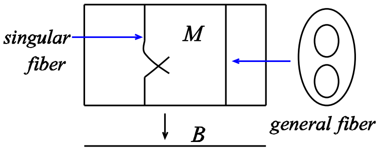
R. E. Gompf proved: A Lefschetz fibration is a symplectic 4-manifold.

Therefore

Symplectic 4-manifols $\fallingdotseq$ Lefschetz fibrations

Generalization of L.F.'s to holomorphic fibrations

M: Complex surface, B: Riemann surface,
A holomorphic map $\varphi : M \to B$ is called a holomorphic fibration (or degenerating family of Riemann surfaces) iff φ is a proper surjective holomorphic map.
General fiber of $\varphi : M \to B$ is a Riemann surface ($\cong \Sigma_g$),
$\exists$ some singular fibers.



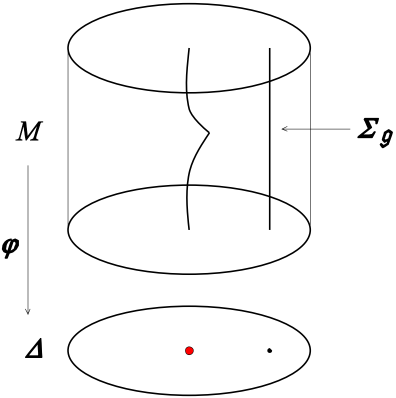
Classification of singular fibers

To study topology of such holomorphic fibrations, we have to start with the local theory, i.e., topological classification of singular fibers:

Let Δ denote the unit disk in $\mathbb{C}$;

$$\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$$

Degenerating family of Riemann surfaces over △ (1)



Over the center, we admit any type of singular fiber.

Degenerating families over Δ (2)

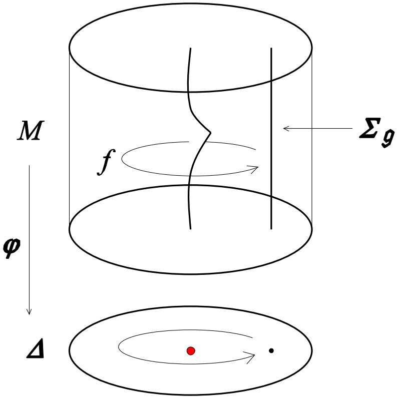
Definition

Two degenerating families $(M_1, \varphi_1, \Delta_1)$ and $(M_2, \varphi_2, \Delta_2)$ are Toplogically equivalent (denoted by $\overset{top}{\cong}$), if $\exists$ orientaion preserving homeomorphisms $H : M_1 \longrightarrow M_2$ and $h : \Delta_1 \longrightarrow \Delta_2$ s.t.

$$\begin{array}{ccc} M_1 & \xrightarrow{H} & M_2 \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ \Delta_1 & \xrightarrow[h]{} & \Delta_2 \end{array}$$

We assume that (M, φ, Δ) is relatively minimal, i.e. fibers do not contain any (-1)-spheres.

Degenerating families over Δ (3)



Topological equivalence class $[M, \varphi, \Delta]$

$\mapsto$ **topological monodromy** $f : \Sigma_g \to \Sigma_g$

Degenerating families over Δ (4)

In the case of Lefschetz type singular fibers, the topological monodromy was a (-1)-Dehn twist about the vanishing cycle.
In general case, topological monodromy belongs to pseudo-periodic maps defined yesterday:

$$[f] : \text{pseudo-periodic} \Longleftrightarrow \begin{cases} [f] : \text{periodic, or} \\ [f] : \text{parabolic.} \end{cases}$$

A parabolic map $[f]$ maight have a fractional Dehn twist about a reducing curve C. $[f]$ is called of negative twist if this twist is negative (with "negative screw numbers "in Nielsen's terminology).

Degenerating families over Δ (5)

Fact: Topological monodromy f is a pseudo-periodic map of negative twist. (Long history: A'Campo, Lẽ, Michel, Weber in Milnor fiberings, and Imayoshi, Shiga-Tanigawa, Earle-Sipe in families of Riemann surfaces)

Theorem (M. and Montesinos 1991/92), Bull. AMS. '94

$$\{(M,\varphi,\Delta)\}/\overset{top}{\cong} \qquad \overset{\longleftrightarrow}{bijection}$$

{pseudo-periodic mapping classes of negative twists}/conj.

$$([M,\varphi,\Delta] \mapsto f : \text{topological monodromy})$$

An interesting point would be

$$\{(M, \varphi, \Delta)\}/ \overset{top}{\cong}$$

$$\underset{bijection}{\longleftrightarrow}$$

{pseudo-per.mappings of negative twists}/conj.

Left-hand side ······ Objects in comlex analysis

Right-hand side ······ Purely topological objects.

Global theory of holomorphic fibrations ?

This is not yet successful.

We would like to change the subject here,
and will consider the problem of constructing the "universal" degenerating family.

Motivation

Riemann surfaces

- **topologically classified by genus g**
- **complex analytically classified by the "moduli space".**

Degenerate Riemann surfaces

- **topologically classified by pseuod-periodic maps**
- **complex analytically calssified by the "compactified moduli space" ("Deligne-Mumford compactification").**

But the last poit seems not yet completely clarified. Our theorem ("hopefully"proved in 2010) gives an exact formulation of this correspondence in terms of "orbifolds".

Recall : Teichmüller space $T(\Sigma_g)$

$T_g = T(\Sigma_g)$ **classifies all the conformal sructures (or complex analytic structures) on** Σ_g **up to isotopy (or equivalently, up to homotopy).**

Teichmüller space (bis)

More precisely:

(S, w): S a Riemann surface, $w : S \to \Sigma_g$ an orientation preserving homeomorphism.

$(S_1, w_1) \sim (S_2, w_2)$: equivalent iff $\exists$ isotopically (or equivalently, homotopically) commutative diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{w_1} & \Sigma_g \\ t\downarrow & & \downarrow = \\ S_2 & \xrightarrow[w_2]{} & \Sigma_g \end{array}$$

where $t : S_1 \to S_2$ is a biregular map (a conformal isomorphism).

Definition. $\boxed{T_g = T(\Sigma_g) = \{(S, w)\}/\sim}$

T_g classifies "marked" Riemann surfaces

(S, w) : a "marked" Riemann surface.

S : a Riemann surface,

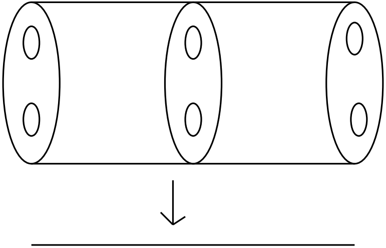
$w : S \longrightarrow \Sigma_g$: a "marking" which topologically identifies S with a fixed topological surface Σ_g.

Bers' tautological family of marked Riemann surfaces

Bers constructed a family of Riemann surfaces (Acta Math. 1973)

$$V(\Sigma_g) \longrightarrow T(\Sigma_g).$$

Over a point $[S, w] \in T_g$, the Riemann surface S is situated.



Recall : Γ_g acts on T_g

Assume $g \geqq 2$. Γ_g acts on T_g:

For $[f] \in \Gamma_g$ and $p = [S, w] \in T_g$, define

$$[f]_*[S, w] = [S, f \circ w]$$

- T_g is a $(3g-3)$-dimensional complex bounded domain (Ahlfors, Bers), and Γ_g acts holomorphically.
- T_g is a metric space (w. "Teichmüller metric"), and Γ_g acts isometrically
- The action of Γ_g on T_g is properly discontinuous.

Moduli space $M(\Sigma_g)$

Moduli space of genus g is defined as :

$$M_g = M(\Sigma_g) = T_g/\Gamma_g.$$

Since the action od Γ_g is properly discontinuous, the moduli space $M_g(= T_g/\Gamma_g)$ is a normal complex space.

Γ_g acts on the fiber space $V(\Sigma_g) \to T(\Sigma_g)$

$\Gamma_g = \Gamma(\Sigma_g)$ acts on $V(\Sigma_g) \to T(\Sigma_g)$ in a fiber preserving manner, (Bers, Acta Math,130, 1973).
By taking the quotient, we get Bers' fiber space over the moduli space $Y(\Sigma_g) \to M(\Sigma_g)$

$$\Gamma_g \quad \text{acts on} \quad V(\Sigma_g) \to T(\Sigma_g)$$

$$\Downarrow \quad \text{quotient}/\Gamma_g$$

$$Y(\Sigma_g) = V(\Sigma_g)/\Gamma_g \to M(\Sigma_g) = T(\Sigma_g)/\Gamma_g$$

M_g parametrizes all Riemann surfaces without marking

- Each Riemann surface S corresponds to a unique point $[S] \in M_g$.
- Over the point $[S] \in M_g$, the Riemann surface S is situated (as a fiber of $Y(\Sigma_g) \to M_g$.)
- If S has a non-trivial symmetry (i.e., $\mathrm{Aut}(S) \neq \{1\}$), then the fiber is $S/\mathrm{Aut}(S)$.

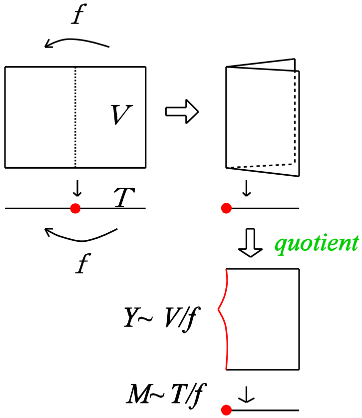
This last degenerate fiber is a singular fiber with periodic monodromy.

Idea

Recall that a singular fiber over Δ was classified by a pseudo-periodic map.

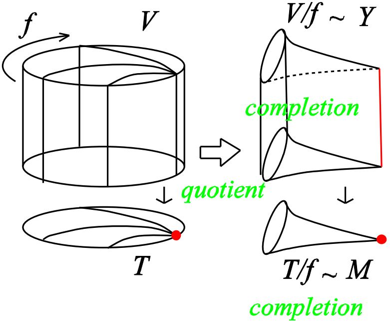
- If the monodromy is periodic, it appears as an inner singular fiber of $Y(\Sigma_g) \to M_g$.
- If the monodromy is parabolic, it will appear as an outer singular fiber on the "boundary" of the "compactified " fiber space $\overline{Y(\Sigma_g)} \to \overline{M(\Sigma_g)}$.

Conceptual explanation: periodic case



singular fiber $\longleftrightarrow$ **periodic monodromy** f

Conceptual explanation: parabolic case



singular fiber $\longleftrightarrow$ **parabolic monodromy** f