New Development of Asymptotic Analysis and Dynamical Systems
(漸近解析と力学系の新展開)

Date: 2009, June 15(Mon) – June 19(Fri.)
Place: RIMS Kyoto University, Room Number 115

Program

June 15 (Monday)

13:30 – 14:30 ŌUCHI, Sunao (大内　忠) (Sophia University)
Multisummability of solutions of formal power series of differential equations I

14:45 – 15:45 YAMAZAWA, Hiroshi (山澤 浩司) (Caritas University)
On construction of true solution with logarithm term and multisummability for some linear partial differential equations

16:00 – 17:00 MIYAKE, Masatake (三宅 正武) (Nagoya University)
Newton polygon and Gevrey hierarchy in index formulas of a singular system of ODOps on formal Gevrey spaces

June 16 (Tuesday)

9:45 – 10:45 YOSHINO, Masafumi (吉野 正史) (Hiroshima University)
Resummation and analytic continuation of an asymptotic solution of a small denominator problem

11:00 – 12:00 ŌUCHI, Sunao (大内 忠) (Sophia University)
Multisummability of solutions of formal power series of differential equations II

13:30 – 14:30 CHIBA, Hayato (千葉 逸人) (Kyoto University)
Fast-slow systems with codimension 2 fold points

14:45 – 15:45 MATSUOKA, Chihiro (松岡 千博) (Ehime University)
Borel-Laplace transform of a difference equation associated with Henon maps (joint work with HIRAIDE, Koichi)

16:00 – 17:00 ITO, Hidekazu (伊藤 秀一) (Kanazawa University)
Generalized action-angle coordinates near singularities of superintegrable systems

18:00 – **banquet** *(The place is to be announced.)*

June 17 (Wednesday)

10:00 – 11:00 COSTIN, Ovidiu (Ohio State University)
Behavior of lacunary series at the natural boundary (joint work with M. Huang)

11:15 – 12:15 COSTIN, Rodica D. (Ohio State University)
Differential systems with Fuchsian linear part: correction and linearization, normal forms and matrix valued orthogonal polynomials

free time in the afternoon

June 18 (Thursday)

9:45 – 10:45	COSTIN, Ovidiu (Ohio State University) *The one-dimensional Schrödinger equation for potential wells and barriers. Borel summation and Gamow vectors. (joint work with M. Huang)*
11:00 – 12:00	SHUDO, Akira (首藤 啓) (Tokyo Metropolitan University) *Role of natural boundaries of KAM curves in dynamical tunneling problems*
13:30 – 14:30	KAMIMOTO, Shingo (神本 晋吾) (University of Tokyo) *On a Schrödinger operator with a merging pair of a simple pole and a simple turning point, I – WKB theoretic transformation to the canonical form*
14:45 – 15:45	KOIKE, Tatsuya (小池 達也) (Kobe University) *On a Schrödinger operator with a merging pair of a simple pole and a simple turning point, II – Computation of Voros coefficients and its consequence*
16:00 – 17:00	UMETA, Youko (梅田 陽子) (Hokkaido University) *Multiple-scale analysis for the first Painlevé hierarchy (joint work with Takashi Aoki and Naofumi Honda)*

June 19 (Friday)

9:30 – 10:30	TAKAHASHI, Kinya (高橋 公也) (Kyushu Institute Technology) *Semiclassical analysis of multi-dimensional barrier tunneling*
10:40 – 11:40	SASAKI, Shinji (佐々木 真二) (RIMS Kyoto University) *On the classification of Stokes graphs for second order Fuchsian-type equations*
11:15 – 12:15	TAKEI, Yoshitsugu (竹井 義次) (RIMS Kyoto University) *Summability of formal solutions of PDEs and the geometry of Stokes curves*
	free discussions in the afternoon

The up-to-date program and the abstracts are also available at

`http://www.math.sci.hiroshima-u.ac.jp/~yoshino/rims0906.html`

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On construction of true solution with logarithm term and multisummability for some linear partial differential equations

Hiroshi YAMAZAWA
Department of Language and Culture, Caritas Junior College

In this tale, let us study a construction of a true solution for some linear partial differential equations by using Theory of Multisummability. For example let us consider the following linear partial differential equation:

$$(t,x) \in \mathbb{C} \times \mathbb{C},$$

$$(0.1) \qquad \big(t\frac{\partial}{\partial t} - \rho(x)\big)u(t,x) = \sum_{j=0}^{2} a_j t\big(t\frac{\partial}{\partial t}\big)^j \big(\frac{\partial}{\partial x}\big)^{2-j} u(t,x) + f(t,x)$$

where

- $\cdot$ $a_j \in \mathbb{C}$ for $j = 0, 1, 2$
- $\cdot$ a function $\rho(x)$ is holomorphic in a neighborhood of $x = 0$ with $Re\rho(0) > 0$, $\rho(0) \notin \mathbb{N} = \{0, 1, \dots\}$
- $\cdot$ a function $f(t,x)$ is holomorphic in a neighborhood of $(t,x) = (0,0)$ with $f(0,x) \equiv 0$.

For this equation, there exists the following formal solution:

$$(0.2) \qquad U(\varphi)(t,x) = \sum_{i=1}^{\infty} u_i(x)t^i + \varphi(x)t^{\rho(x)} + t^{\rho(x)} \sum_{\substack{k \le 2i \\ i \ge 1}} \varphi_{i,k}(x)t^i(\log t)^k$$

where $\{u_i(x)\}_{i\ge1}$, $\varphi(x)$ and $\{\varphi_{i,k}(x)\}_{k\le 2i, i\ge 1}$ are holomorphic functions in a common neighborhood of $x = 0$. Moreover $\varphi(x)$ is any holomorphic function and a function $\varphi_{i,k}(x)$ depends on $\varphi(x)$. For this formal solution $U(\varphi)$, we have that

$$(0.3) \qquad \sum_{i=1}^{\infty} \frac{u_i(x)}{\Gamma(i)}t^i + \varphi(x)t^{\rho(x)} + t^{\rho(x)} \sum_{\substack{k \le 2i \\ i \ge 1}} \frac{\varphi_{i,k}(x)}{\Gamma(i)}t^i(\log t)^k$$

is converges for a sufficiently small t.

In the case $a_0 \neq 0$ or $a_1 \neq 0$, we can show that we have a true solution with the asymptotic expansion $U(\varphi)$ by [O1] (Ōuchi '95).

At next we consider the case $a_0 = a_1 = 0$ and $a_2 \neq 0$. If we take $\varphi(x) \equiv 0$, then we have $\varphi_{i,k}(x) \equiv 0$ for all (i,k), that is, the formal solution $U(0)$ becomes a formal power series solution.

For this case, we have a true solution by using Theory of Multisummability by [O2] (Ōuchi '02). Many mathematicians study Theory of Summability; W. Balser, M. Miyake, Y. Sibuya and M. Hibino and so on.

In this talk in the case $\varphi(x) \not\equiv 0$ we study a construction of a true solution with an asymptotic expansion $U(\varphi)$ by using Theory of Multisummability. In fact, the formal function

$$\varphi(x)t^{\rho(x)} + t^{\rho(x)} \sum_{\substack{k \leq 2i \\ i \geq 1}} \varphi_{i,k}(x) t^i (\log t)^k \tag{0.4}$$

satisfies the equation

$$\big(t\frac{\partial}{\partial t} - \rho(x)\big)u(t,x) = \sum_{j=0}^{2} a_j t \big(t\frac{\partial}{\partial t}\big)^j \big(\frac{\partial}{\partial x}\big)^{2-j} u(t,x). \tag{0.5}$$

Then as generally we will unfold Theory of Summability for the following equation:

$$\big(t\frac{\partial}{\partial t} - \rho(x)\big)\phi(t,x) = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(t,x) \big(t\frac{\partial}{\partial t}\big)^j \big(\frac{\partial}{\partial x}\big)^{\alpha} \phi(t,x) \tag{0.6}$$

where $(t,x) \in \mathbb{C} \times \mathbb{C}^d$ and $a_{j,\alpha}(t,x)$ is holomorphic in $(t,x) = (0,0)$ with $a_{j,\alpha}(0,x) \equiv 0$.

Suppose that it states in detail at the time in a lecture.

References

[1] Ōuchi, S., Genuine solutions and formal solutions with Gevrey type estimates of nonlinear partial differential equations, *J. Math. Sci. Univ Tokyo* **1** (1995), 375–417.

[2] ———, Multisummability of Formal Solutions of Some Linear Partial Differential Equations, *J. Diff. Equ.* **185** (2002), 513–549.

Newton polygon and Gevrey hierarchy in index formulas of a singular system of ODOps on formal Gevrey spaces

Masatake MIYAKE (Nagoya University)

1. Introduction

We study a system of ordinary differential operators $L \equiv L(z, D)$, which is singular at $z = 0$, of the following form,

$$(1) \qquad L(z,D) = z^{p+1}D - A(z), \quad A(z) = (a_{ij}(z)) \in M_N(\mathcal{R}\{z\}),$$

where $p \in \mathbb{N} := \{0, 1, 2, 3, \cdots\}$, $z \in \mathbb{C}$, $D = d/dz$ and $M_N(\mathcal{R}\{z\})$ denotes the set of $N \times N$ matrices with entries of meromorphic functions at $z = 0$ which we denote by $\mathcal{R}\{z\}$.

The purpose of this study is to define the Newton polygon $N(L)$ of the system $L(z, D)$, and to prove an index formula on formal Gevrey space which is explicitly calculated from the Newton polygon $N(L)$. This gives an extension of the theory for a single operator by J.-P. Ramis [Ram] to a system of operators.

List of Notations :

1. $\mathbb{C}[z]$, $\mathbb{C}\{z\}$, $\mathbb{C}[[z]]$: polynomials, convergent series, formal poer series of z over $\mathbb{C}$, respectively.
2. $\mathcal{R}[z] := \mathbb{C}[z]/\mathbb{C}[z] = \{f(z)/g(z)\ ;\ f(z), g(z) \in \mathbb{C}[z]\}$, $\mathcal{R}\{z\} := \mathbb{C}\{z\}/\mathbb{C}\{z\}$, $\mathcal{R}[[z]] := \mathbb{C}[[z]/\mathbb{C}[[z]]$: : rational functions, meromorphic functions, formal meromorphic functions which may have pole singularities at $z = 0$, respectively.
3. $M_N(R)$, $GL_N(R)$: $N \times N$-matrices, invertible matrices of entries in a ring R, respectively.

2. Review of Newton polygon and index formula by J.-P. Ramis

We give a short review of Newton polygon for a single operator and an index formula on formal Gevrey space proved by J.-P. Ramis [Ram].

Let a single operator $P \equiv P(z, D)$ be given by

$$(2) \qquad P = a_0(z)(zD)^m + \sum_{j=1}^{m} a_j(z)(zD)^{m-j}, \quad a_j(z) \in \mathbb{C}\{z\},\ a_0(z) \not\equiv 0.$$

We denote by $O(a_j) = r_j \in \mathbb{N} \cup \{+\infty\}$ the order of zeros at $z = 0$, and consider the following correspondence between an operator and a figure,

$$a_j(z)(zD)^{m-j} \quad \longleftrightarrow \quad Q(m-j, r_j) := \{(x, y) \in \mathbb{R}^2\ ;\ x \le m-j,\ y \ge r_j\}.$$

Then the Newton polygon $N(P)$ is defined by

$$(3) \qquad N(P) := \text{Convex} - \text{hull} \left\{ \bigcup_{j=0}^{m} Q(m-j, r_j) \right\}.$$

The Newton polygon $N(P)$ is characterized by its vertexes which we denote by $\{(m-j_i, r_{j_i})\}_{i=0}^{k}$ $(k \geq 0)$ with

$$0 = j_0 < j_1 < \cdots < j_k \leq m, \quad r_{j_0} > r_{j_1} > \cdots > r_{j_k}. \tag{4}$$

Let the slopes of $k+2$ sides of $N(P)$ denoted by $\{\rho_i\}_{i=-1}^{k}$ be given by

$$\rho_i = \frac{r_i - r_{i+1}}{j_{i+1} - j_i}, \qquad \rho_{-1} := \infty > \rho_0 > \rho_1 > \cdots > \rho_{k-1} > 0 =: \rho_k. \tag{5}$$

We note that the operator $P(z, D)$ is regular singular at $z = 0$ if and only if $k = 0$.

J.-P. Ramis showed that these verteces and slopes characterize the operator $P(z, D)$ on formal Gevrey space $\mathcal{G}^s$ $(1 \leq s \leq \infty)$, where $\mathcal{G}^s \subset \mathcal{C}[[z]]$ is defined by

$$u(z) = \sum\nolimits_{n=0}^{\infty} u_n z^n \in \mathcal{G}^s \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \sum\nolimits_{n=0}^{\infty} u_n \frac{\zeta^n}{(n!)^{s-1}} \in \mathbb{C}\{\zeta\} \quad (1 \leq s < \infty), \tag{6}$$

and $\mathcal{G}^\infty := \mathbb{C}[[z]]$.

Then he proved that the mapping $P(z, D) : \mathcal{G}^s \to \mathcal{G}^s$ defines a Fredholm operator for every $s \in [1, \infty]$, and its index $\chi(P; \mathcal{G}^s) := \dim_{\mathbb{C}} \ker(P; \mathcal{G}^s) - \dim_{\mathbb{C}} \operatorname{coker}(\mathrm{P}; \mathcal{G}^s)$ is obtained as follows: The index $\chi(P; \mathcal{G}^s)$ is a right continuous step function on $s \in [1, +\infty]$ with k discontinuous points $\{s_i\}_{i=0}^{k-1}$ given by

$$s_i = 1 + 1/\rho_i, \qquad s_{-1} := 1 < s_0 < s_1 < \cdots < s_{k-1} < +\infty =: s_k, \tag{7}$$

and the index is given by

$$\chi(P; \mathcal{G}^s) = -r_{j_i}, \quad s_{i-1} \leq s < s_i, \ (0 \leq i \leq k). \tag{8}$$

When $i = k$ the formula holds for $s_{k-1} \leq s \leq s_k = +\infty$.

Remark 1. The index formula was proved by Malgrange [Mal] in cases $s = 1$ and ∞, and by Komatsu [Kom] for a normalized system of higher order operators in case $s = 1$. For a general system of operators, the formula was proved in a joint paper [M-Y] with M. Yoshino in case $s \in \mathbb{R}$ for operators with polynomial coefficients, where the determinant theory of matrices of differential operators on formal Gevrey space was employed. We note that the restriction of polynomial coefficients is removed for $1 \leq s \leq \infty$ by compact perturbation argument.

3. Statement of Results

We define the order of a meromorphic function as follows. For a non zero formal meromorphic function $a(z) \in \mathcal{R}[[z]]$,

$$O(a) := r \in \mathbb{Z} \quad \Longleftrightarrow \quad a(z) = z^r b(z), \ b(z) \in \mathbb{C}[[z]], \ b(0) \neq 0. \tag{9}$$

and if $a(z) = 0$ we define $O(a) := +\infty$. The order $O(A)$ of a matrix function $A(z)$ is defined similarly.

Def.3.1 (Volevič's weight) For a matrix function $A(z) = (a_{ij}(z)) \in M_N(\mathcal{R}[[z]])$, let $r_{ij} = O(a_{ij}) \in \mathbb{Z} \cup \{+\infty\}$. Then Volevič's weight $V(A) \in \mathbb{Q} \cup \{+\infty\}$ is defined by

$$V(A) := \min_{1 \leq n \leq N} \ \min_{1 \leq i_1 < \cdots < i_n \leq N} \ \min_{\sigma \in S_n} \ \frac{1}{n} \sum\nolimits_{k=1}^{n} r_{i_k, i_{\sigma(k)}}. \tag{10}$$

Def.3.2 (Full rank system of irregular singular type) A system $L = z^{p+1}D - A(z)$ $(A(z) \in M_N(\mathcal{R}[[z]])$ is called a full rank system of irregular type if

$$V(A) < p, \qquad O(\det A(z)) = NV(A). \tag{11}$$

The definition implies for the characteristic polynomial $\det(\lambda I - A(z)) = \lambda^N - \sum_{j=1}^{N} a_j(z)\lambda^{N-j}$ that

$$O(a_j) \geq jV(A), \quad O(a_N) = NV(A),$$

but the converse does not hold.

Def.3.3 (Regular singular type) A system L is called of regular singular type if $V(A) \geq p$.

Theorem A (Reduction to a canonical form) *A singular system L with $A(z) \in M_N(\mathcal{R}\{z\})$ is reduced into a canonical form by a formal meromorphic transformation as follows: $\exists\, P(z) \in GL_N(\mathcal{R}[[z]])$, $\exists\, N = N_1 + \cdots + N_k + N_{k+1}$ $(N_j \geq 0, k \geq 0)$ such that*

$$P^{-1}(z)L(z,D)P(z) = z^{p+1}D - \mathrm{Triang}(A_1(z), \cdots, A_k(z), A_{k+1}(z)) \tag{12}$$

where $\mathrm{Triang}(\cdots)$ *denotes a block wise diagonal matrix with j-th diagonal block $A_j(z)$, where*

$$L_j(z,D) := z^{p+1}D - A_j(z), \quad A_j(z) \in M_{N_j}(\mathcal{R}[[z]]),$$

is full rank system of irregular singular type for $1 \leq j \leq k$ and $L_{k+1}(z,D)$ is of regular singular type with a property

$$V(A_1) < V(A_2) < \cdots < V(A_k) < p \leq V(A_{k+1}). \tag{13}$$

Def.3.4 (Newton polygon) For a singular system L with $A(z) \in M_N(\mathcal{R}\{z\})$, the Newton polygon $N(L)$ is defined by

$$N(L) := N(\det(L_1(z,\zeta)) + \cdots + N(\det L_k(z,\zeta)) + N(\det L_{k+1}(z,\zeta)). \tag{14}$$

We note that the Newton polygon $N(p)$ for a polynomial $p(z,\zeta) = \sum_{j=0}^{m} a_j(z)(z\zeta)^j$ is defined similarly with an operator $p(z,D)$.

Theorem B (Index formula) *A singular system L with $A(z) \in M_N(\mathbb{C}\{z\})$ defines a Fredholm operator on $\mathcal{G}^s$ $(1 \leq s \leq \infty)$, and the index $\chi(L;\mathcal{G}^s)$ is obtained similarly with a single operator by using the Newton polygon $N(L)$.*

Remark 2. In a recent paper with K. Ichinobe [M-I], we defined and characterized the irregularity of solutions of $L(z,D)u(z) = 0$ as a maximal rate of exponential growth when $z \to 0$, which we denoted by $\rho(L)$. Now we know that this problem is equivalent to characterize the steepest slope of the Newton polygon $N(L)$. The arguments developed in the paper play important and powerful roles in this study.

References

[Kom] Komatsu, H., J. Fac. Sci. Univ. Tokyo Sect. IA **18** (1971), 379-398. [Mal] Malgrange, B., Enseignement Math. **20** (1970), 146-176. [M-I] Miyake, M. and Ichinobe, K., Funkcial. Ekvac., **52** (2009), 53 - 82 . [M-Y] Miyake, M. and Yoshino, M., Funkcial. Ekvac., **38** (1995), 329-342. [Ram] Ramis, J.-P., Mom. Amer. Math. Soc., **48** (1984).

Fast-slow systems with codimension 2 fold points

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Abstract. The existence of stable periodic orbits and chaotic invariant sets of singularly perturbed problems of fast-slow type with codimension two fold points is proved by means of the boundary layer technique and the blow-up method. In particular, the blow-up method is effectively used for analyzing the flow near the fold points in order to show that an orbit near the fold points is extended along a solution of the first Painlevé equation in the blow-up space.

1 Introduction

Singularly perturbed ordinary differential equations of the form

$$\left\{\begin{array}{l} \dot{x}_1 = f_1(x_1, \cdots, x_n, y_1, \cdots, y_m, \varepsilon), \\ \quad\vdots \\ \dot{x}_n = f_n(x_1, \cdots, x_n, y_1, \cdots, y_m, \varepsilon), \\ \dot{y}_1 = \varepsilon g_1(x_1, \cdots, x_n, y_1, \cdots, y_m, \varepsilon), \\ \quad\vdots \\ \dot{y}_m = \varepsilon g_m(x_1, \cdots, x_n, y_1, \cdots, y_m, \varepsilon), \end{array}\right. \tag{1.1}$$

are called a *fast-slow system*, where the dot ($\dot{}$) denotes the derivative with respect to time t, and where $\varepsilon > 0$ is a small parameter. The unperturbed system of this system is given by

$$\left\{\begin{array}{l} \dot{x}_1 = f_1(x_1, \cdots, x_n, y_1, \cdots, y_m, 0), \\ \quad\vdots \\ \dot{x}_n = f_n(x_1, \cdots, x_n, y_1, \cdots, y_m, 0), \\ \dot{y}_1 = 0, \\ \quad\vdots \\ \dot{y}_m = 0. \end{array}\right. \tag{1.2}$$

The set of fixed points of the unperturbed system is called a *critical manifold*, which is defined by

$$\mathcal{M} = \{(x_1, \cdots, x_n, y_1, \cdots, y_m) \in \mathbf{R}^{n+m} \mid f_i(x_1, \cdots, x_n, y_1, \cdots, y_m, 0) = 0,\ i = 1, \cdots, n\}. \tag{1.3}$$

Typically $\mathcal{M}$ is an m-dimensional manifold. It is known that the shape and stability of the critical manifold determine global behavior of the flow of the fast-slow system.

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2 Main results

In this talk, we investigate the 3-dimensional fast-slow system of the form

$$\begin{cases} \dot{x} = f_1(x, y, z, \varepsilon), \\ \dot{y} = f_2(x, y, z, \varepsilon), \\ \dot{z} = \varepsilon g(x, y, z, \varepsilon), \end{cases} \tag{2.1}$$

with a small parameter $\varepsilon > 0$. Note that the critical manifold

$$\mathcal{M} = \{(x, y, z) \in \mathbf{R}^3 \mid f_1(x, y, z, 0) = f_2(x, y, z, 0) = 0\} \tag{2.2}$$

gives curves on $\mathbf{R}^3$ in general. Under some assumptions for the shape and stability of the critical manifold (see Fig.1), we will prove that there exists a stable periodic orbit if $\varepsilon > 0$ is sufficiently small. Further, we will prove that if ε increases, a succession of the period-doubling bifurcations occurs and it induces a chaotic invariant set.

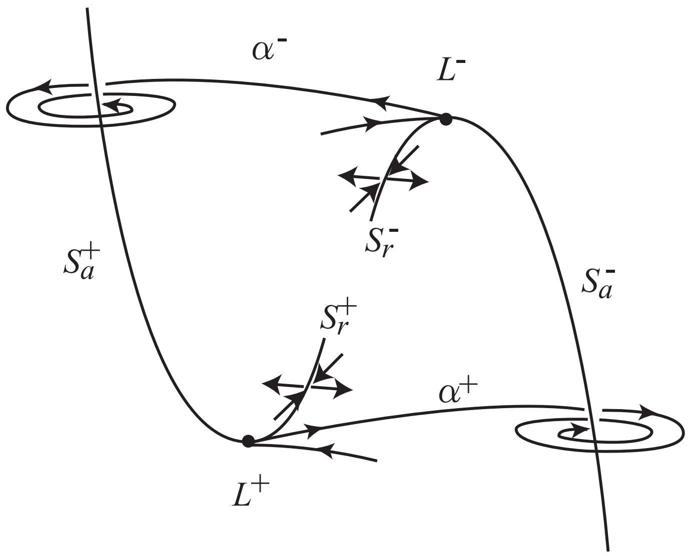

Fig. 1 Critical manifold $\mathcal{M} = S_a^+ \cup S_r^+ \cup S_a^- \cup S_r^-$ and the flow of the unperturbed system of Eq.(2.1).

To show the existence of a stable periodic orbit or a chaotic invariant set, we have to calculate a succession of the transition maps (Poincaré maps) along the flow of Eq.(2.1) as Fig.2. It is shown that if $\varepsilon > 0$ is sufficiently small, the global Poincaré map is contractive and thus a stable periodic orbit exists (Fig.3(a)). On the other hand, if $\varepsilon > 0$ increases, the global Poincaré map proves to have Smale horseshoe and thus the existence of a chaotic invariant set is proved (Fig.3(b)).

When calculating the transition maps near the fold points $L^\pm$ of the critical manifold, the blow-up method is effectively used in order to “de-singularize” the fold points. It is shown that in the blow-up space, a solution orbit of the system (2.1) is extended along a solution of the first Painlevé equation.

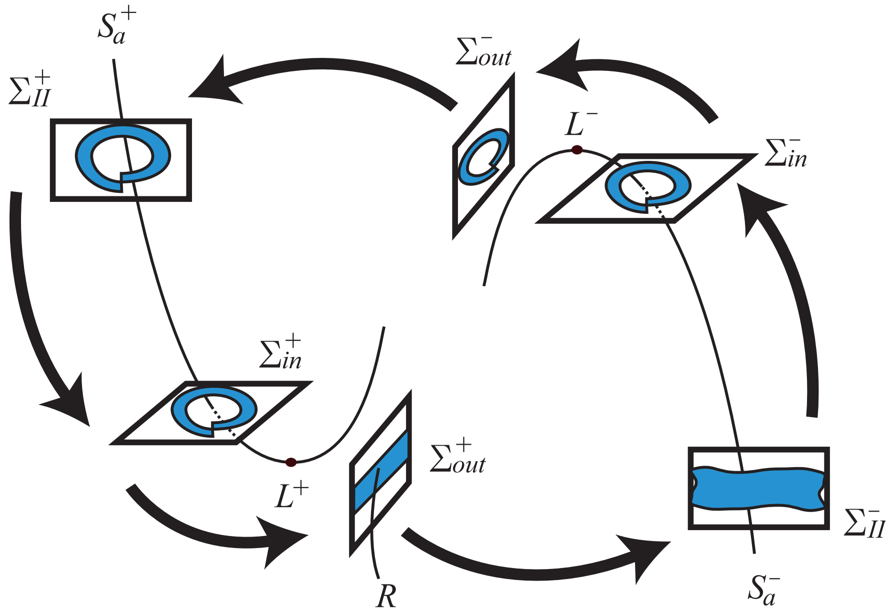

Fig. 2 Poincaré sections and a schematic view of the images of the rectangle R under a succession of the transition maps.

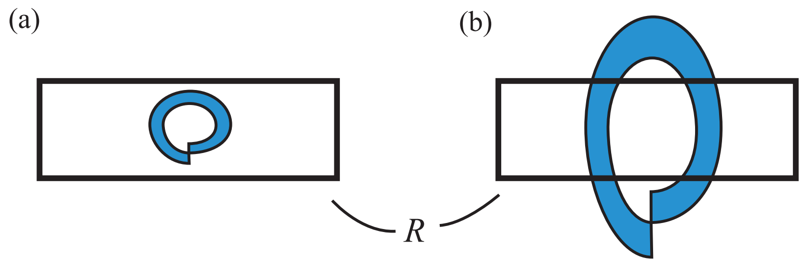

Fig. 3 Positional relationship between the rectangle R and its image under the Poincaré map.

References

[1] H. Chiba, Fast-slow systems with codimension 2 fold points, (submitted; http://yang.amp.i.kyoto-u.ac.jp/~chiba/fast_slow1.pdf)

Speaker : Rodica D. Costin

Affiliation : Ohio State University

Title : *Differential systems with Fuchsian linear part: correction and linearization, normal forms and matrix valued orthogonal polynomials*

Abstract :

Differential systems with a Fuchsian linear part are studied in regions including all the singularities in the complex plane of these equations. Such systems are not necessarily analytically equivalent to their linear part (they are not linearizable) and obstructions are found as a unique nonlinear correction after which the system becomes formally linearizable. More generally, normal forms are found.

Linearizability of differential equations is closely related to integrability.

The corrections and the normal forms of are generated constructively, using expansions in sequences of matrix-valued polynomials which turn out to have many of the properties associated to classical orthogonal polynomials.

Definition of orthogonality for the classical Jacobi polynomials for general weights will also be discussed.

Speaker： COSTIN, Ovidiu
Affiliation： Ohio State University
Title： *Behavior of lacunary series at the natural boundary* (joint work with M. Huang)

Abstract：

We develop a local theory of lacunary Dirichlet series of the form $\sum_{k=1}^{\infty} c_k \exp(-zg(k))$, $g' \to \infty$ as z approaches $i\mathbb{R}$. These series occur in many applications in Fourier analysis, infinite order differential operators, number theory and holomorphic dynamics among others. We obtain blow up rates in measure along the imaginary line and asymptotic information at $z = 0$.

When sufficient analyticity information on g exists, we obtain Borel summable expansions at points on the boundary, giving exact local description. The singular behavior has remarkable universality and self-similarity features.

The Bötcher map at infinity of polynomial iterations of the form $x_{n+1} = \lambda P(x_n)$, $|\lambda| < \lambda_0(P)$, turns out to have uniformly convergent Fourier expansions in terms of simple lacunary series. For the quadratic map $P(x) = x - x^2$, $\lambda_0 = 1$, and the Julia set is the graph of this Fourier expansion in the main cardioid of the Mandelbrot set.

The Borel summable transseries gives a detailed description of Julia sets of classes of polynomial maps.

Speaker: COSTIN, Ovidiu
Affiliation: Ohio State University
Title: *The one-dimensional Schrodinger equation for potential wells and barriers. Borel summation and Gamow vectors.* (joint work with M. Huang)

Abstract:
We analyze the detailed time dependence of the wave function $\psi(x,t)$ for one dimensional Hamiltonians $H = -D^2 + V(x)$ where $V(x)$ (modeling barriers or wells) and $\psi(x,0)$ are compactly supported. We show that the dispersive part of ψ (its asymptotic series in powers of $t^{-1/2}$) is Borel summable. The remainder, the difference between $\psi(x,t)$ and the Borel sum, is a convergent expansion of the form

$$\sum(g_k \Gamma_k(x) e^{-\gamma_k t})$$

where the functions $\Gamma_k(x)$ are the Gamow vectors of H, and the $-\gamma_k$ are the associated resonances. Generically, all coefficients $g_k \in \mathbb{C}$ are nonzero. For large k, γ_k is proportional to $k \log k + ik^2 Pi^2/4$.

(Gamow vectors are poles of the analytically continued Green's function, and they are generalized eigenfunctions of the Hamiltonian, with "purely growing" conditions at infinity.)

After Borel summation, the expansion is very rapidly convergent allowing a sharp qualitative and quantitative control on the wave function for moderate or large time.

The analytic structure of is perhaps surprising: in general (even in simple examples such as square wells), $\psi(x;t)$ is given by a lacunary series.ψ turns out to be C^∞ in t but nowhere analytic on $R+$ which is a natural boundary of ψ.

Summability of formal solutions of PDEs and the geometry of Stokes curves

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Let us consider the following initial value problem for a partial differential equation in two complex variables $(t, z) \in \mathbb{C}$:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - a(z)\dfrac{\partial^2}{\partial z^2}\right) u(t,z) = 0, \\ u(0,z) = \varphi(z). \end{cases} \tag{1}$$

As one can readily observe, the initial value problem (1) has a solution of the form

$$\hat{u}(t,z) = \sum_{j=0}^{\infty} \left(a(z)\frac{d^2}{dz^2}\right)^j \varphi(z)\, \frac{t^j}{j!}, \tag{2}$$

which is in general divergent for an analytic initial value $\varphi(z)$. More generally, such a formal (divergent) solution $\hat{u}(t,z)$ also exists for the following equation as well:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - P(z, \dfrac{\partial}{\partial z})\right) u(t,z) = 0, \\ u(0,z) = \varphi(z). \end{cases} \tag{3}$$

Since the pioneering work of Lutz-Miyake-Schäfke for the heat equation ([LMS]), many works have been done on the summability of the formal solution $\hat{u}(t,z)$ for these initial value problems. See, e.g., [B], [M], [CT], [BL], ... In these works the case of partial differential equations with constant coefficients are mainly discussed.

In the case of equations with variable coefficients, on the other hand, very few things are known even for such a simple equation as (1). In this talk, through the investigation of several typical concrete equations, we would like to show that the geometry of Stokes curves is related to the summability of $\hat{u}(t,z)$ in the variable coefficients case. To be more specific, we apply the Laplace transformation $\mathcal{L}_{t\to\tau}$ for the variable t to Equation (1) and employ the scaling $\tau = \eta^2$ to obtain a one-dimensional Schrödinger equation

$$\left(\frac{\partial^2}{\partial z^2} - \eta^2 \frac{1}{a(z)}\right)\psi(z,\eta) = 0 \quad \text{with} \quad \psi = \mathcal{L}u. \tag{4}$$

Then we see that the WKB solutions and the Stokes geometry of (4) play an important role in the study of the resummation of the formal solution $\hat{u}(t,z)$ of the original partial differential equation (1). The details will be discussed in the talk.

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Resummation and analytic continuation of an asymptotic solution of a small denominator problem

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1. Linearization problem. Let $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$, $n \geq 2$ be the variable in $\mathbb{C}^n$. We consider a semi-simple holomorphic singular vector field $\mathcal{X} = \sum_{j=1}^n a_j(y)\frac{\partial}{\partial y_j}$, $a_j(0) = 0$ $(j = 1, \ldots, n)$ defined in some neighborhood of the origin of $\mathbb{C}^n$. The following equations give the change of the coordinates $y = x + v(x)$ which linearizes $\mathcal{X}$

$$\text{(H)} \qquad \varepsilon^{-1}\mathcal{L}v_j = \lambda_j v_j + R_j(x + v(x)),\ \mathcal{L} = \sum_{j=1}^n \lambda_j x_j \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, n,$$

where $R = (R_1, \ldots, R_n)$ is the nonlinear part of $\mathcal{X}$, $R = O(|x|^2)$ when $|x| \to 0$.

We can uniquely construct a singular perturbative solution (SP-solution in short) of (H) $v(x, \varepsilon)$ in the form

$$v(x, \varepsilon) = \sum_{\nu=0}^{\infty} \varepsilon^{-\nu} v_\nu(x) = v_0(x) + \varepsilon^{-1}v_1(x) + \cdots,$$

where the sum with respect to ε is a formal sum, and $v_\nu(x)$ are vector-valued functions which are holomorphic in some neighborhood of the origin independent of ν.

2. Borel-Laplace resummation. We use the Borel-Laplace transform with respect to ε in order to construct an exact solution of (H) although (H) is a semilinear equation. Let the direction ξ $(0 \leq \xi < 2\pi)$ and the opening $\theta > 0$ are given. We define $S_{\xi,\theta}$ by $S_{\xi,\theta} = \left\{\varepsilon \in \mathbb{C};\ |\arg \varepsilon - \xi| < \frac{\theta}{2}\right\}$. Then we have

Theorem 2. (Resummation) . *Assume that the Poincaré condition or the following condition is satisfied.*

$$\text{(0.1)} \qquad \exists\ t,\ 0 \leq t \leq 2\pi,\ e^{it}\lambda_j \in \mathbb{R} \setminus \{0\} \quad (j = 1, 2, \ldots, n).$$

Then there exist a ξ, $\theta > 0$, a neighborhood U of the origin $x = 0$ and $V(x, \varepsilon)$ such that $V(x, \varepsilon)$ is holomorphic in $(x, \varepsilon) \in U \times S_{\xi,\theta}$ and it gives a solution of (H). The SP-solution $v(x, \varepsilon)$ is the G^2-asymptotic expansion of $V(x, \varepsilon)$ in $U \times S_{\xi,\theta}$, when $\varepsilon \to \infty$. Namely, for each $N \geq 1$ and $R > 0$, there exist $C > 0$ and $K > 0$ such that

$$\text{(0.2)} \qquad \left| V(x, \varepsilon) - \sum_{\nu=0}^{N} \varepsilon^{-\nu} v_\nu(x) \right| \leq CK^N N! |\varepsilon|^{-N-1},$$

for all $(x, \varepsilon) \in U \times S_{\xi,\theta}$, $|\varepsilon| \geq R$.

3. Analytic continuation of SP-solution. We now study the analytic continuation with respect to ε of the resummed SP-solution, and we apply it to the study of asymptotic property of divergent series which appears in solving (H) in a small denominator case.

First we study the analytic continuation in the Poincaré case. We recall that there exist an infinite number of ε's on the right half-plane $\Re \varepsilon > 0$ for which the resonace occurs. For the sake of simplicity we call these values of ε ε-resonances. These values of ε accumulate only at infinity in the Poincaré case, and may accumulate on the real line in other cases by the assumption (0.1). The resummed SP-solution $V(x, \varepsilon)$ may be singular with respect to ε at these values in view of the proof of Theorem 2. Our first observation is that the analytic continuation (with respect to ε) of the SP-solution to the right half-plane is identical with the classical Poincaré series solution. More precisely we have

Theorem 3. *Suppose that the Poincaré condition is verified. Then $V(x,\varepsilon)$ is analytically continued as a single-valued function of ε to a non ε-resonant point along any bounded path which does not meet with the ε-resonances . If $\varepsilon = 1$ is nonresonant, then the analytic continuation of the resummed SP-solution to $\varepsilon = 1$ coincides with the classical Poincaré series solution of (H).*

Now we study the solvability of (H) without assuming a Diophantine condition. Let σ be a nonnegative integer and Γ be an open connected neighborhood of the origin, and let $0 \le c < \pi/2$. We define $H_\sigma \equiv H_{\sigma,c,\Gamma}$ as the set of holomorphic (vector) funtions $v(\zeta) = (v_1(\zeta), \dots, v_N(\zeta))$ of $\zeta = \eta + i\xi \in \Gamma + i\mathbb{R}^n$ such that

$$\|v\|_{\sigma,c,\Gamma} := \sup_{\eta\in\Gamma} \int_{\mathbb{R}^n} \langle\zeta\rangle^\sigma e^{c|\xi|} |v(\zeta)| d\xi < \infty, \tag{0.3}$$

where $\langle\zeta\rangle = 1 + \sum_{j=1}^n |\zeta_j|$, $|\xi| = |\xi_1| + \cdots + |\xi_n|$, and $|v(\zeta)| = (\sum_{j=1}^N |v_j(\zeta)|^2)^{1/2}$. The space $H_{\sigma,c,\Gamma}$ is a Banach space with the norm (0.3). We define the multi sector S_c by

$$S_c := (S_0)^n, \quad S_0 = \{z \in \mathbb{C}; z = re^{i\theta}, |\theta| < c, r > 0\}. \tag{0.4}$$

Let $f(x)$ be an integrable N- vector function on $\mathbb{R}_+^n$, $\mathbb{R}_+ := \{t \in \mathbb{R}; t \ge 0\}$ and let $\hat{f}(\zeta)$ be the Mellin transform of f

$$\hat{f}(\zeta) \equiv M(f)(\zeta) = \int_{\mathbb{R}_+^n} f(x) x^{\zeta - e} dx, \quad e = (1, \dots, 1),\ \zeta = \eta + i\xi, \eta \in \Gamma, \xi \in \mathbb{R}^n, \tag{0.5}$$

where $x^{\zeta-e} = x_1^{\zeta_1 - 1} \cdots x_n^{\zeta_n - 1}$, $\zeta = (\zeta_1, \dots, \zeta_n)$. It is easy to see that $\hat{f}(\zeta)$ is analytic if the integral (0.5) absolutely converges. The inverse Mellin transform is given by

$$f(x) = M^{-1}(\hat{f})(x) = (2\pi i)^{-n} \int_{\mathbb{R}^n} \hat{f}(\eta + i\xi) x^{-\eta - i\xi} d\xi, \tag{0.6}$$

where η is so taken that the integral converges. We note that if $\hat{f} \in H_{\sigma,c,\Gamma}$, then the integral (0.6) is a holomorphic function of x in S_c. We note that these formulas follow from the corresponding ones of the Fourier-Laplace transform by the change of variables $e^{i\theta_j} \to x_j$.

We define $\mathcal{H}_{\sigma,c,\Gamma}$ as the inverse Mellin transform of $H_{\sigma,c,\Gamma}$. We note that the Mellin transform gives the one to one correspondence between the spaces $\mathcal{H}_{\sigma,c,\Gamma}$ and $H_{\sigma,c,\Gamma}$. For $u \in \mathcal{H}_{\sigma,c,\Gamma}$ we define the norm $\|u\|_{\sigma,c,\Gamma}$ of u by

$$\|u\|_{\sigma,c,\Gamma} := \|M(u)\|_{\sigma,c,\Gamma}.$$

For an integer $k \ge 1$ we denote by $(\mathcal{H}_{\sigma,c,\Gamma})^k$ the product of k copies of $\mathcal{H}_{\sigma,c,\Gamma}$. The norm in $(\mathcal{H}_{\sigma,c,\Gamma})^k$ is defined as the sum of the norms of each component. For simplicity, we denote the norm in $(\mathcal{H}_{\sigma,c,\Gamma})^k$ by $\|\cdot\|_{\sigma,c,\Gamma}$ if there is no fear of confusion.

Let Γ_0 be a connected neighborhood of the origin in $\mathbb{R}^n$. Let $\Gamma_1 \subset -\mathbb{R}_+^n$ be an open connected convex cone with vertex at the origin such that $\langle\lambda, \eta\rangle \ne 0$ for every $\eta \in \Gamma_1$, $\eta \ne 0$. Let $\vec{\Gamma}(\zeta) := \Gamma(\zeta_1) \cdots \Gamma(\zeta_n)$, where $\Gamma(z)$ is the Gamma function. Let $\hat{R}(\zeta)$ be the

Mellin transform of $R(\zeta)$. We assume that the nonlinear term $R_j(x)$ $(j = 1, 2, \dots , n)$ has the form

$$\hat{R}_j(\zeta) = \sum_{\alpha\in\mathbb{Z}_+^n} r_\alpha^j(\zeta)\vec{\Gamma}(\zeta+\alpha) \quad (j = 1, 2, \dots , n). \tag{0.7}$$

Here $r_\alpha^j(\zeta)$ is an entire function such that there exists $K \geq 1$ satisfying that $|r_\alpha^j(\zeta)| \leq e^{-K|\xi|}/|\alpha|^{|\alpha|}$ when $\xi = \Im\zeta \to \infty$.

Example. Let $\alpha = (\alpha_1, \dots , \alpha_n)$, $\alpha_k > 0$ and let c_j be given constants. If $R_j(x)$ $(j = 1, 2, \dots , n)$ are given by $R_j(x) = \sum_{\alpha,\text{finite}} c_{j,\alpha}x^\alpha \exp(-x_1 - \cdots - x_n)$, then the condition (0.7) holds.

Let $0 < \theta_0 < \pi/2$ be a given constant, and let $\tau_0 = \pm\pi/2$. We define the sector Σ_{τ_0,θ_0} with the vertex at $\varepsilon = 1$, the direction τ_0 and opening θ_0 by

$$S_{\tau_0,\theta_0} = \{\varepsilon \in \mathbb{C}; |arg(\varepsilon - 1) - \tau_0| < \theta_0/2\}. \tag{0.8}$$

Let $V(x, \varepsilon)$ be the analytic continuation of the resummed SP-solution given in Theorem 2. Let $\Im\,\varepsilon \neq 0$ and let us, for the moment, assume that we can expand

$$V(x, \varepsilon) = \sum_\alpha V_\alpha(\varepsilon)x^\alpha \tag{0.9}$$

in the convergent series of x at $x = 0$. We note that the radius of convergence of the series may tend to zero when $\varepsilon \to 1$ in the case where all λ_j's are real numbers. Then we have

Theorem 5. *Suppose that the λ_j's in (H) are real numbers and nonresonant, $\lambda_j - \langle\lambda, \alpha\rangle \neq 0$ for all $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geq 2$, $j = 1, 2, \dots , n$. Assume that (0.7) is satisfied. Then there exist $K_0 > 0$ and an integer $k_0 > 0$ such that if $\|R\|_{k_0,c,\Gamma_0} < K_0$ and $\|\nabla R\|_{k_0,c,\Gamma_0} < K_0$, then there exists $u(x, \varepsilon)$ holomorphic in $(x, \varepsilon) \in S_c \times S_{\tau_0,\theta_0}$ such that, for every $N = 0, 1, 2, \dots$, there exists $R_N(x, \varepsilon)$ being holomorphic in $(x, \varepsilon) \in S_c \times S_{\tau_0,\theta_0}$ and satisfying $R_N(x, \varepsilon) = O(|x|^{N+1})$ as $|x| \to 0$, $(x, \varepsilon) \in S_c \times S_{\tau_0,\theta_0}$ such that*

$$u(x, \varepsilon) - \sum_{|\alpha|\leq N} V_\alpha(\varepsilon)x^\alpha = R_N(x, \varepsilon), \quad (x, \varepsilon) \in S_c \times S_{\tau_0,\theta_0}. \tag{0.10}$$

Borel-Laplace transform of a difference equation associated with Henon maps

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A polynomial diffeomorphism $f : \mathbb{C}^2 \to \mathbb{C}^2$

(1)
$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 + y \quad ax^2 \\ bx \end{pmatrix}$$

is called the (complex) *Hénon map* , where $a \neq 0$ and $b \neq 0$ are complex numbers, and fixes the following two points:

$$\left(\frac{b \quad 1 \pm \sqrt{(b \quad 1)^2 + 4a}}{2a}, \quad \frac{b \quad 1 \pm \sqrt{(b \quad 1)^2 + 4a}}{2a} b \right).$$

Let $P = (x_f, y_f)$ be one of them, and let $\alpha = \alpha_1$ be an eigenvalue of the derivative Df_P at P. Then we have the quadratic equation

(2)
$$\alpha^2 \quad \lambda\alpha \quad b = 0,$$

where $\lambda = \quad 2ax_f$, and another eigenvalue α_2 of Df_P coinsides with $\lambda \quad \alpha = \quad b/\alpha$. As usual, define the *stable* and *unstable* manifolds at P by

$$\begin{aligned} W^s(P) &= \{Q \in \mathbb{C}^2 | f^n(Q) \to P \text{ as } n \to \infty\}, \\ W^u(P) &= \{Q \in \mathbb{C}^2 | f^n(Q) \to P \text{ as } n \to \quad \infty\} \end{aligned}$$

respectively. It is well-known that if P is a saddle point, i.e. $0 < |\alpha_1| < 1$ and $|\alpha_2| > 1$, then $W^s(P)$ is actually an analytic submanifold of $\mathbb{C}^2$ which is an injective immersion of the complex plane $\mathbb{C}$, and tangent to the eigenspace for α_1 in the tagent space $T_P\mathbb{C}^2$ at P, and the similar fact for $W^u(P)$ holds.

We construct a novel function in order to describe the stable and unstable manifolds of the Hénon map which is constructed by the use of Borel-Laplace transform.

Let α be one of eigenvalues of the derivative Df_P of the Hénon map f at a fixed point $P = (x_f, y_f)$. We consider an f-invariant curve at $P = (x_f, y_f)$ parameterized by the complex variable $t \in \mathbb{C}$ as follows:

$$t \mapsto \begin{pmatrix} x(t) + x_f \\ y(t) + y_f \end{pmatrix} = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix},$$

such that

$$f : \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \mapsto \begin{pmatrix} X(t+1) \\ Y(t+1) \end{pmatrix} = \begin{pmatrix} 1 + Y(t) \quad aX(t)^2 \\ bX(t) \end{pmatrix}.$$

Replacing $X(t)$ and $Y(t)$ as $x(t)$ and $y(t)$ again, the following difference equation of the second kind is obtained

(3)
$$x(t+1) \quad \lambda x(t) \quad bx(t \quad 1) = \quad a\{x(t)\}^2,$$

together with $y(t) = bx(t \quad 1)$.

In order to solve (3), we express $x(t)$ with the Laplace integral on some Riemann surface X;

(4)
$$x(t) = \mathcal{L}[X](t) = \int_\gamma \mathrm{e}^{-\zeta t} X(\zeta) \mathrm{d}\zeta$$

where the contour γ is chosen depending on the positions and the form of branch points of X. Substituting (4) into (3), we obtain an integral equation for $X(\zeta)$:

$$AX = \quad aX * X + C, \tag{5}$$

where $C(\zeta)$ is an entire function of exponential type, $A(\zeta) = \mathrm{e}^{-\zeta} \quad \lambda \quad b\mathrm{e}^{\zeta}$, and $\;*\;$ denotes the convolution defined by

$$F * G = \int_0^{\zeta} F(\zeta \quad \zeta')G(\zeta')\mathrm{d}\zeta'.$$

Setting

$$X(\zeta) = a_0 + \tilde{X}(\zeta), \tag{6}$$

and substituting (6) into (5), we have

$$A\tilde{X} + 2aa_0 * \tilde{X} = W. \tag{7}$$

where $W = W_0 \quad a\tilde{X} * \tilde{X}$ and $W_0 = \quad aa_0^2\zeta \quad a_0A + C$.

We expand $\tilde{X}(\zeta)$ and $W(\zeta)$ with a formal parameter σ as

$$\tilde{X}(\zeta) = \sum_{n=1}^{\infty} \sigma^n \tilde{X}_n(\zeta), \quad W(\zeta) = \sum_{n=0}^{\infty} \sigma^{n+1} W_n(\zeta).$$

Substituting these into (7), we have the solution $\tilde{X}_n$ $(n = 1, 2, \quad)$

$$\tilde{X}_n = A^{-1}F_0 \int_0^{\zeta} \frac{W'_{n-1}}{F_0} \mathrm{d}\zeta' \quad (n = 1, 2, \dots), \tag{8}$$

where $W'_{n-1} = \frac{\mathrm{d}W_{n-1}}{\mathrm{d}\zeta}$.

Let $\alpha \neq 0$ be one of eigenvalues of the derivative Df_P at P. We define the lattice Γ_α generated by $\quad \log|\alpha|$ as follows. For $k \in \mathbb{Z}$, let $\zeta_k = \rho + (2k\pi + \theta)\mathrm{i} \quad \zeta_{k1}$, where

$$\rho = \quad \log|\alpha|, \qquad \pi < \theta = \arg \alpha \quad \pi,$$

and let

$$\Gamma_\alpha = \{\zeta \in \mathbb{C} \mid \zeta = \sum_{l=1}^{N} \zeta_{kl}, \quad \zeta_{kl} = l\rho + (2k\pi + \theta)\mathrm{i}, \quad N = 1, 2, \quad \}. \tag{9}$$

It is easy to see that Γ_α is on the right half plane of $\mathbb{C}$ if $0 < |\alpha| < 1$, on the left half plane if $|\alpha| > 1$, and on the imaginary axis if $|\alpha| = 1$. Note that Γ_α is dense in the imaginary axis in the case of $|\alpha| = 1$ (see Figure 1).

Lemma 1 *Let $|\alpha| \neq 1$. For $\zeta \in \mathbb{C} \setminus \Gamma_\alpha$ and a path ω from the origin to ζ in $\mathbb{C} \setminus \Gamma_\alpha$, there is a smooth path δ from the origin to ζ homotopic to ω in $\mathbb{C} \setminus \Gamma_\alpha$ such that $\zeta/2 \in \delta$ and δ is symmetrical with respect to $\zeta/2$.*

Theorem 1 *Let $\zeta = \zeta_{kN} + \xi$ $(|\xi| < \rho,\ k \in \mathbb{Z})$ and δ be a path from the origin to ζ in $\mathbb{C} \setminus \Gamma_\alpha$. Then the solution $\tilde{X}(\zeta) = \tilde{X}(\zeta, \delta)$ to Eq. (7) is given as the limit of a Riemann suface $\tilde{X}^{(N)}(\zeta)$:*

$$\tilde{X}(\zeta) = \lim_{N \to \infty} \tilde{X}^{(N)}(\zeta), \quad \tilde{X}^{(N)}(\zeta) = \sum_{n=1}^{\infty} \tilde{X}_n^{(N)}(\zeta).$$

The convolution $W_{n-1}^{(N)}(\zeta)$ and the solution $\tilde{X}_n^{(N)}(\zeta)$ $(N \quad 2)$ are given by

$$W_{n-1}^{(N)}(\zeta_{kN} + \xi) = \begin{cases} \sum_{m=0}^{\infty} v_{n-1,m+n-1}^{(N)} \xi^{m+n-1} (\log\xi)^n + \mathrm{reg}^{(n-1)}(\xi), & (1 \quad n \quad N \quad 1) \\ \sum_{m=0}^{\infty} v_{n-1,m+n-1}^{(N)} \xi^{m+n-1} (\log\xi)^N + \mathrm{reg}^{(N-1)}(\xi), & (n \quad N) \end{cases}$$

$$\tilde{X}_n^{(N)}(\zeta_{kN} + \xi) = \begin{cases} \sum_{m=0}^{\infty} b_{n,m+n-1}^{(N)} \xi^{m+n-1} (\log\xi)^n + \mathrm{reg}^{(n-1)}(\xi), & (1 \quad n \quad N \quad 1) \\ \sum_{m=0}^{\infty} b_{n,m+n-1}^{(N)} \xi^{m+n-1} (\log\xi)^N + \mathrm{reg}^{(N-1)}(\xi), & (n \quad N) \end{cases}$$

(10)

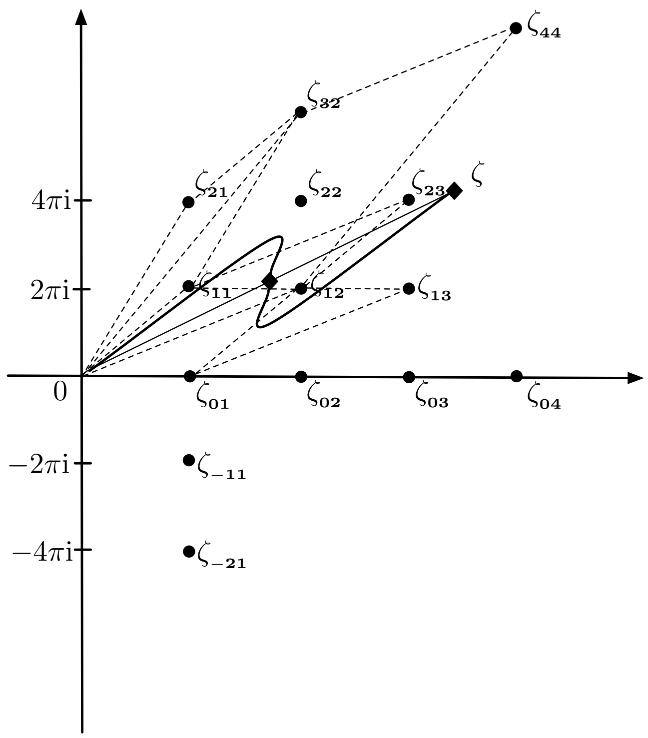

Figure 1: Lattice Γ_α and path δ (thick solid curve), where we set $0 < |\alpha| < 1$ and $\theta = 0$ in ζ_{k1}.

where $v^{(N)}_{n-1,m+n-1}$ *and* $b^{(N)}_{n,m+n-1}$ *are complex coefficents which do not depend on the choice of the vertical index* k *in the* N*-th singularity. The notation* $\mathrm{reg}^{(n-1)}$ *is given by*

$$\mathrm{reg}^{(n-1)}(\xi) = \sum_{m=0}^{n-1} R_m(\xi)(\log\xi)^m,$$

where $R_m(\xi) = * + *\xi + *\xi^2 + \quad$ $(m = 0, 1, 2, \quad)$ *is a regular function with complex coefficients* $*$*'s.*

From now on, we drop the index k and rewrite the N-th singularity ζ_{kN} as ζ_N. Thus, we obtain the solution $\tilde{X}^{(N)}(\zeta)$ for the N-th singularity

$$\begin{aligned} \tilde{X}^{(N)}(\zeta) &= \sum_{n=1}^{\infty} \tilde{X}^{(N)}_n(\zeta) \\ &= \sum_{n=N}^{\infty} \sum_{m=0}^{\infty} b^{(N)}_{n,m+n-1}(\zeta \quad \zeta_N)^{m+n-1} \left[\log(\zeta \quad \zeta_N)\right]^N + \mathrm{reg}^{(N-1)}(\zeta \quad \zeta_N). \end{aligned} \tag{11}$$

For the coefficient $b^{(N)}_{n,m+n-1}$ in the solution $\tilde{X}^{(N)}_n$ for the N-th singularity $\zeta = \zeta_N$ $(N \quad 1)$, the following lemma holds.

Lemma 2 *There exist constants* $C > 0$ *and* $K > 0$ *which depend on the eigenvalue* α*, and the first coefficient* $b^{(N)}_{n,n-1}$ *and the higher order coefficient* $b^{(N)}_{n,m+n-1}$ $(m = 0, 1, 2, \quad)$ *of the* n*-th convolution for* $n \quad N$ *in the* N*-th solution* $\tilde{X}^{(N)}_n(\zeta_N + \xi)$ *are estimated as*

$$\begin{aligned} |b^{(N)}_{n,n-1}| &\quad C^n \frac{|\alpha|^{N\log N}}{n!}, \\ |b^{(N)}_{n,m+n-1}| &\quad K^n |b^{(N)}_{n,n-1}|. \end{aligned}$$

for all $n \quad N$ $(N \quad 1)$.

Semiclassical analysis of multi-dimensional barrier tunneling

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Quantum tunneling for one dimensional systems is well captured by the established semiclassical method, i.e., instanton[1], which is also applicable, ignoring rigorousness of mathematics, to multi-dimensional tunneling if a system under consideration is integrable. However, this is not the case for non-integrable systems, because invariant manifolds formed by first integrals, i.e, tori, are partially or completely broken in the phase space for non-integrable systems and instanton that is a periodic (or regular) trajectory of imaginary time evolution loses the guiding manifold, i.e., torus extended into the complex domain. Actually, there observe numerically and experimentally many complicated tunneling phenomena, which can not be explained by the instanton theory. In recent years, many authors have made significant contributions to progress in the semiclassical description of tunneling for multi-dimensional systems[2].

Two-dimensional barrier systems are the most simplified class of systems which exhibit tunneling inherent in multi-dimensional systems and most of them are non-integrable in the sense of Painlevé test. Two different semiclassical mechanisms may work on two-dimensional barrier tunneling, i.e., the well-established instanton mechanism and the recently discovered mechanism utilizing complexified stable-unstable manifolds as the guide of tunneling paths[3]. The new mechanism is explained as follows: on the potential barrier there always exists an unstable periodic orbit (PO) dividing the transmitted side from the incident side. The PO is accompanied by the manifolds, called stable (unstable) manifold, on which all the trajectories approach to (separate from) the PO. The stable manifold W_s of PO always intersects with the initial manifold $\mathcal{I}$ supporting the incident quantum state, if both manifolds are extended into the complex phase space. There exist complex tunneling trajectories starting from a neighborhood of the intersection and approaching exponentially the real phase space along the stable and unstable manifolds. It was also confirmed that this tunneling mechanism also works in the case that chaos exists in the real space[4].

The new mechanism, namely stable-unstable manifold guided tunneling (called SUMGT for brevity), rules tunneling, when the contribution of SUMGT trajectories overcomes instanton. Actually SUMGT violates instanton mechanism in this case. There is another interesting regime in some ranges of parameters, in which both mechanisms, instanton and SUMGT, simultaneously contribute to tunneling, though SUMGT makes less contributions. So the mechanism ruling the tunneling process changes between the two regimes. In this talk, we first clarify the difference between instanton and SUMGT from the view points of complexified classical dynamics and complexified semiclassical method, and demonstrate how the transition between the two mechanisms occurs making a remarkable change in the spectrum of tunneled particles.

The model system we study is given by

$$H(Q,P,\omega t) = \frac{1}{2}P^2 + (1+\epsilon \sin \omega t)\mathrm{sech}^2 Q. \tag{1}$$

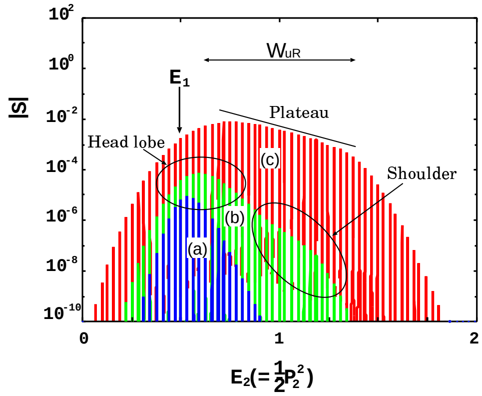

Figure 1: Tunneling spectra (absolute value of the S-matrix) at three representative values of the perturbation strength: (a) $\epsilon = 0.1$, (b) $\epsilon = 0.2$, and (c) $\epsilon = 0.4$. E_2 is the output energy and the input energy is chosen as $E_1 = 0.5$. $\omega = 0.3$ and $\hbar = 1000/(3\pi \times 2^{10}) \sim 0.1036$.

Here a plane wave with a constant input energy $E_1 = 0.5$ is incident. For a strong perturbation at $\epsilon = 0.4$, the spectrum envelop forms a plateau spread over a wide range of energy, whose width corresponds to oscillating range of real unstable manifold W_{uR} at an asymptotic side ($|Q| \gg 1$). This characteristic spectrum is the result of SUMGT. On the other hand, for a weak perturbation at $\epsilon = 0.1$, the spectrum is localized around E_1, for which instanton must be available. The interesting case appears for an intermediate strength at $\epsilon = 0.2$. The spectrum seems to be constructed by the superposition of two characteristic spectra: a head lobe will be explained by perturbed instanton theory, while a wide shoulder over an upper range of energy will be formed by SUMGT. Thus, the two tunneling mechanisms will coexist in this case. We explore what kind of change occurs with increase of ϵ from the semiclassical point of view.

The semiclassical S-matrix is given by [5]

$$S(E_2, E_1) \sim \lim_{Q_1, |Q_2| \to \infty} \sum_{\text{c.t.}} \frac{\sqrt{|P_2||P_1|}}{\sqrt{2\pi i \hbar P_1 P_2}} \sqrt{-\frac{\partial^2 S_S}{\partial E_1 \partial E_2}} \times e^{-i(P_2 Q_2 - P_1 Q_1)/\hbar} e^{\frac{i}{\hbar} S_S(Q_2, E_2, Q_1, E_1)}, \quad (2)$$

where $S_S = \int_{Q_1}^{Q_2} P dQ - \int_{t_1}^{t_2} H(Q, P, \omega t) dt + E_2 t_2 - E_1 t_1$ is the classical action. The summation $\sum_{\text{c.t.}}$ is taken over all the contributing trajectories satisfying the input and final boundary conditions [5]. The coordinate Q_i and momentum P_i (or energy$E_i = P_i^2/2$) at the input side ($i = 1$) and at the output side ($i = 2$) are observed quantities and should be taken as real values, whereas times t_i are unobserved and can take *complex* variables. We can regard (Q, P) as functions of the lapse time $s \equiv t - t_1$ ($\in \mathbf{C}$), initial time t_1 ($\in \mathbf{C}$) and the set of fixed initial values (Q_1, P_1) ($\in \mathbf{R^2}, Q_1 > 0$). Then, the initial manifold is defined by $\mathcal{I} = \{(t_1, Q, P) | t_1 \in \mathbf{C}, Q = Q_1, P = P_1\}$.

The intersection t_{1c} of $\mathcal{I}$ with W_s in the complex domain is obtained by using Melnikov-type method and the imaginary part of t_{1c} is given by

$$\mathrm{Im}t_{1c} = \frac{1}{\omega}\cosh^{-1}\left\{\frac{1-E_1}{\epsilon(1-\chi(\omega))}\right\}, \tag{3}$$

where $\chi(\omega)$ is defined by $\chi(\omega) \equiv 2\omega\int_0^\infty \frac{\sin\omega s}{1+e^{2\sqrt{2}s}}\,\mathrm{d}s$. Then, the imaginary depth of the critical point $\mathrm{Im}t_{1c}$ decreases with increase of ϵ. There always exist SUMGT trajectories starting in a small neighborhood of t_{1c}. On the other hand, the imaginary time evolution of the instanton is estimated by $t_{inst} = -i\pi/\sqrt{2E_2}$.

Comparison of $\mathrm{Im}t_{1c}$ with $\mathrm{Im}t_{inst}$ gives the criterion to judge which semiclassical mechanism, instanton or SUMGT, dominates the tunneling process. If the condition $\mathrm{Im}t_{1c} > 1.5|\mathrm{Im}t_{inst}|$ is well satisfied, the barrier-penetrated tunneling, namely instanton, dominates the tunneling process, otherwise the critical point t_{1c} destroys the instanton mechanism and only tunneling trajectories of SUMGT which go over close to the barrier top contribute to forming a plateau spectrum. However even if the above condition is satisfied, the trajectories of SUMGT still survive and contribute to forming a shoulder part of the spectrum.

Further, combining the Melnikov-type method with a low-frequency approximation allows exploring more detail properties of SUMGT trajectories starting from the neighborhood of t_{1c}. It is very important to consider geometric structure of stable and unstable manifolds extended into the complex domain, which guide the SUMGT trajectories. Movable singularities of the classical solution exhibiting anomalous movement on stable and unstable manifolds also play a key role. Details will be discussed in the talk.

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